BOOK REVIEWS

Linear operators. Part II. *Spectral theory*. By Nelson Dunford and Jacob T. Schwartz, with the assistance of William G. Bade and Robert G. Bartle. Interscience, New York, 1963. Pp. 859–1923. $35.

If a mathematical visitor from another century were to visit these shores and inquire about the present state of mathematical analysis, he would not miss much if he limited himself to leafing through the pages of this book. With the long awaited appearance of the second volume, we now see before our eyes a panoramic view, rich in colorful detail, of the whole output of a school of mathematical analysis that started with the work of Volterra and Fréchet near the turn of the century, and reached its peak in Poland, Hungary and the Soviet Union as well as at Chicago and Yale in the thirties and forties.

It is impossible to do justice to this volume by merely listing the contents, as has become customary in a review; we shall instead highlight some of the most valuable and typical parts, referring to Halmos's detailed review of the first volume for a description of the techniques of presentation of the material.

The guiding idea of the entire work is the spectral theory of a single linear operator, and its varied applications. Algebras of operators, specifically B^*-algebras, enter only in an ancillary function as aids in the proof of the spectral theorem, or as convolution algebras. This is all to the good, in view of the emphasis on concrete applications and special linear operators. It is rumored that von Neumann algebras and group representations are to be treated in detail in a separate volume by J. T. Schwartz anyway.

Chapter X, containing a proof of the spectral theorem, and probably written at an early stage, should be read by whoever wants to get the flavor of the exposition. The style is matter-of-fact in a pleasant no-nonsense way; it is a great improvement over certain excesses of rationalistic presentation with which we have become all too familiar. The overwhelming use of intricate notation, also an endemic vice of the day, is carefully avoided; one can open the book at almost any page and easily figure out the meaning of all symbols without having to trace it backwards ad infinitum. Theorems are seldom stated in their full stratospheric generality, but rather at that level where the typical application is to be found. A happy instance of this wise limitation is the treatment of multiplicity theory and spectral representation, in both Chapter X and Chapter XII; here the assumption of separability is a welcome simplification; the proofs, here as elsewhere, are based upon the natural, at times even pedestrian approaches, avoiding clever mystification and elegance at all costs, in favor of sound motivation.

In these days of extreme fragmentation of fields, when some mathematicians take pride in not knowing the applications of their own work, the authors' constant concern with the "Zusammenhang" between theory and application, and between distinct branches of mathematics, accounts for the unusual length of this treatise. This concern is quite apparent in the awesome Chapter XI, bearing the understating title "Miscellaneous Applications." A partial list of the topics treated here might make a good list of headings for as many courses in analysis, as follows: compact groups (including Haar measure and the Peter-Weyl theorem); almost periodic functions; convolution algebras (Plancherel's theorem and neighboring results); closure theorems of Wiener type; spectral synthesis (several detailed results); Hilbert-Schmidt operators, including a beautiful and little-known inequality of Carleman bounding the growth of the resolvent of a Hilbert-Schmidt operator, and some profound and largely original results on the completeness of the eigenfunctions of non-self-adjoint Hilbert-Schmidt operators; the Hilbert-transform theory of Calderón and Zygmund; a detailed previously unpublished study of compact operators, classified according to their trace classes; subdiagonalization of compact operators, including some of the deep results on invariant subspaces initiated by the Russian school. The section on "Notes and Remarks" at the end of the chapter gives, among other tidbits, proofs of the Pontrjagin duality theorem, the Marcinkiewicz interpolation theorem, with recent refinements due to Hörmander, the Paley-Littlewood inequality, as well as thorough references to the literature for topics touched upon but not fully treated in the text, such as eigenvalue distribution for compact operators and various results on topological groups. Finally, two lengthy sections of exercises, written, as elsewhere throughout the book, in the interlocking Pólya-Szegö style, provide drill in special Tauberian theorems, various results on Fourier integrals and singular integrals, location of eigenvalues for finite matrices (work of A. Brauer, inequalities of H. Weyl, Perron-Frobenius theory, Fredholm theory, etc.). The conscientious student may supplement these exercises with the set of 53 given in the following chapter, "Unbounded Selfadjoint Operators" (including unitary semigroups and moment problems); here he will find all the folklore on unbounded operators, dilations of operators, von Neumann's spectral sets, various versions of the Heisenberg uncertainty principle, comparison theorems for completeness of Paley-Wiener type, etc., etc.

Chapter XIII is the central part of the book; it is a complete and definitive treatise on the spectral theory of ordinary selfadjoint differential operators, centering upon the Coddington-Kodaira-Krein-Levinson-Titchmarsh-Weyl theory of the spectral representation. The results on location of the continuous spectrum and the limit point-limit circle dichotomy are a complete survey of work done in the field up to 1958, *literally*. Those results that are not proved in the text are summarized in a lengthy section on notes and remarks, and in a set of 128 exercises which will tax the most conscientious student. A final section of this chapter, 25 pages long, contains specific techniques for computation of spectral resolutions of hypergeometric differential operators. The results derived here are a great improvement over the only treatment previously available on the subject by the late E. C. Titchmarsh, and will be invaluable to the worker in group representations, as well as to any physicist with no more than the average physicist's deficiencies in mathematics.

The final Chapter XIV, on linear partial differential operators, displays, in the words of the authors, "a bouquet of applications of functional analysis." It would be hard to find a bouquet of similar fragrance in the literature: Sobolev's theorem, the theory of distributions (admirably done without the largely irrelevant locally convex frills), the elliptic boundary value problem, including the remarkable Browder completeness theorem, not previously included in a textbook to the reviewer's knowledge, and finally the Friedrichs-Lax-Phillips theory of symmetric hyperbolic systems.

Whoever has followed this book since its beginning, and has derived from it much of his mathematical education and inspiration, as the reviewer has and as a whole generation of analysts will, cannot forget the immense amount of human effort and loving care that went into its composition. As one reads through these pages, the entire world of present-day mathematical analysis springs to life. Since Picard and Goursat, there has not been a textbook on such a gigantic scale. The authors have successfully accomplished a task that many would have regarded as impossible.

To have a nearly complete view of an entire field, that of functional analysis, available in a coherent exposition (which, we hope, will eventually reach several thousand pages with the publication of the remaining $n-2$ volumes), is a benefit that few mathematical disciplines have ever enjoyed—one thinks of Fricke-Klein or Russell-

Whitehead as feeble precedents; inevitably it will have a decisive influence upon the future development of all analysis, as well as of much theoretical physics and other neighboring disciplines.

No serious student of analysis can afford to ignore this treatise. More than our thanks, the authors deserve to be read by anyone who is interested in learning the results of a whole age of mathematics.

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