THE GENUS OF K_n, $n=12s$

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Introduction. This note solves the problem of obtaining a triangular imbedding of K_n, $n=12s$, $s=1, 2, 3, \cdots$ in an orientable 2-manifold. Thus a simple computation shows that the genus of K_n is $(n-3)(n-4)/12$. A solution to the problem was obtained for $s=2^m$, $m=0, 1, 2, \cdots$ in [2] and will be used here. For each m the problem is solved for $s=2^m(2r+1)$, $r=0, 1, 2, \cdots$, though an explicit solution is presented here only for the case $m=0$ and $r\geqq 0$. For general m and r the solution is handled in an analogous fashion. (A general reference in this connection is [3], but see [1] for the methods employed.)

The technique, as indicated in [2], involves expeditious matching of a group to the geometry of a network.

The group. The solution for $r=0$, $m=0$ given in [2] involved a group $G(k)$ of order $3\cdot 2^k$ where $k=(m+2)$. The group elements other than the identity e were represented by

$$(1)\qquad \begin{aligned} &s_i; && i = 1, \cdots, (2^k-1),\\ &t_i, t_i^{-1}; && i = 1, \cdots, 2^k, \end{aligned}$$

where

(2) The s_i are all the elements of order 2,

(3) t_1 and t_{2^k} are elements of order 3,

(4) The indexing is such that

$$\begin{aligned} &\text{(a)}\ s_i\cdot t_i = t_{i+1}, && i = 1, \cdots, (2^k-2),\\ &\text{(b)}\ t_i\cdot s_i = t_{i+1}, && i = 2^k-1. \end{aligned}$$

The group employed in imbedding K_{12s}, $s=2^m(2r+1)$ is $G(m+2)\times Z_{2r+1}$.

The discussion is now restricted to the case $m=0$.

The quotient network. The network consists of r "boxes" and one "crown" and is shown in skeleton form in Figure 1.

Details for the box numbered i are found in Figure 2 if $i\equiv r \bmod 2$, and otherwise in Figure 3. The crown is shown in Figure 4.

The first coordinates of the currents are from the collection (1) augmented by the identity e. The second coordinates are defined as follows

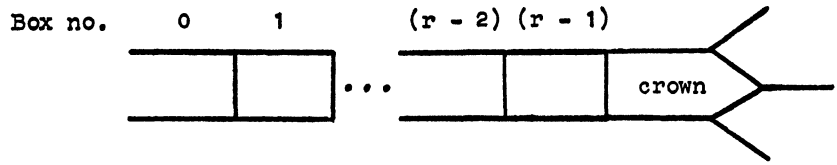


FIGURE 1

$$(5)\qquad \begin{aligned} x_i &= -i; & i &= 0, \cdots, r, \\ y_i &= i + 1; & i &= 0, \cdots, (r-1), \\ z_i &= 2i + 1; & i &= 0, \cdots, r, \\ w_i &= 2i + 2; & i &= 0, \cdots, (r-1). \end{aligned}$$

Notice that

$$(6)\qquad \begin{aligned} x_i + z_i &= y_i; & i &= 0, \cdots, (r-1), \\ x_r + z_r &= x_r, & & \\ y_i - w_i &= x_{i+1}; & i &= 0, \cdots, (r-1). \end{aligned}$$

Figures 2, 3 and 4 have certain nodes indicated by large dots. The rotation at such nodes is clockwise; at all other nodes the rotation is counterclockwise.

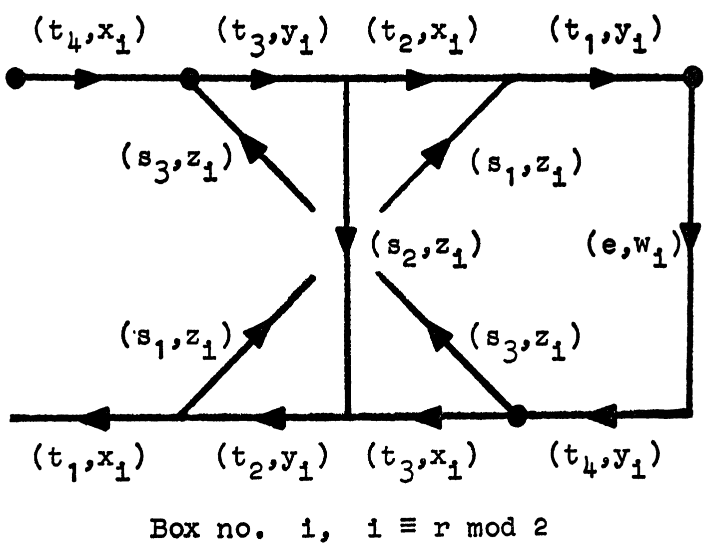
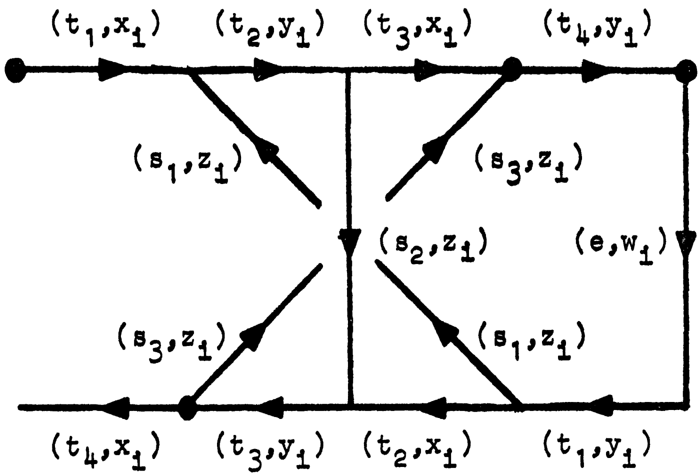


FIGURE 2



Box no. i, $i \not\equiv r \bmod 2$

FIGURE 3

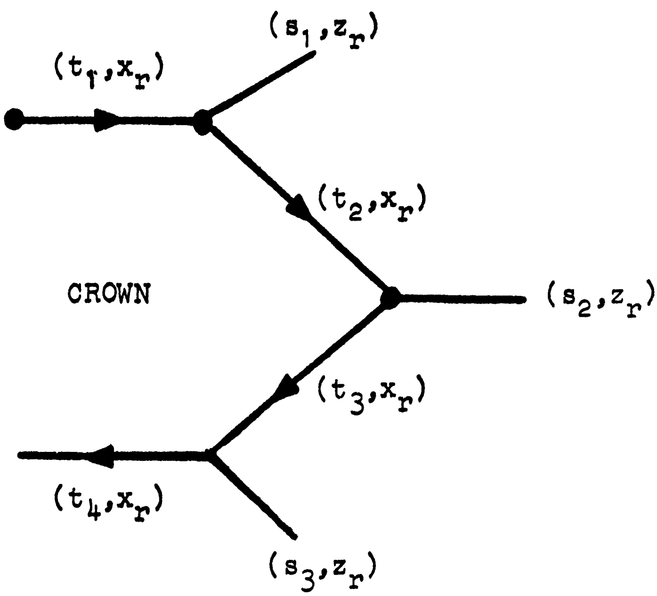


FIGURE 4

For each r, the rotation induces a circulation with a single circuit; hence the index is 1. All nodes are of order 1 or 3. There are three "singular arcs" in the network (all found on the crown) and two "knobs"—the leftmost arcs (see [2]).

So much for the geometry of the network. As to the currents:

1. Because of (1) and (5) each element of $G(2)\times Z_{2r+1}$ except the identity $(e, 0)$ appears exactly once in the circuit, for a total of $(24r+11)$ currents.

2. By (2) the only elements of order 2 are $(s_i, 0)$, $i=1, 2, 3$ and these are the currents found on the three singular arcs.

3. There are two knobs, and these carry currents $(t_1, 0)$ and $(t_4, 0)$, both elements of order 3 by (3).

4. At each node of order 3, by (4) and (6), the product of the *outward* directed currents in cyclic order of rotation is the identity $(e, 0)$.

Thus a triangular imbedding has been achieved. (See the concluding remarks in [2].)

Bibliography

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