AN ANNIHILATOR ALGEBRA WHICH IS NOT DUAL[1]

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1. **Introduction.** The purpose of this note is to give an example of an annihilator algebra which is not dual; no other such example has been published. Here we construct a semi-simple, normed, annihilator algebra which has a closed two-sided ideal which is not an annihilator algebra. Every dual algebra is an annihilator algebra by definition, and every closed ideal in a semi-simple dual algebra is a dual algebra by a theorem of Kaplansky ([**2**, Theorem 2, p. 690] or (iii) in the text of this note). Noting these facts, it follows that the example we construct is not a dual algebra.

Whether every closed two-sided ideal in an annihilator algebra was necessarily an annihilator algebra had been a question of long standing.

The example given here is a normed algebra. The algebra is a Q-algebra (see [**4**, p. 373]), but not, however, a Banach algebra in the given norm. Therefore these questions remain open for the special case of a Banach algebra.

2. **The example.** Let l^p be the algebra of p-summable complex sequences with multiplication performed coordinate-wise. Set $A_1 = l^1$, $A_2 = l^2$ and $A = A_1 \oplus A_2$ (the direct sum of A_1 and A_2). For $x \in A$, we shall write $x = (x_1, x_2)$, where $x_1 \in A_1$, $x_2 \in A_2$. $x_1(i)$ and $x_2(i)$ will denote the ith coordinate of x_1 and x_2 in l^1 and l^2, respectively.

We shall define a norm on A such that A is an annihilator algebra, but not dual, in the topology of this norm.

First we define, for $x \in A$,

$$p(x) = \left(\sum_{i=1}^{\infty} | x_1(i) |^2 + \sum_{i=1}^{\infty} | x_2(i) |^2 \right)^{1/2}.$$

Note that $p(x)$ is a norm on A.

Secondly, since l^1 is properly contained in l^2, we may choose a non-zero linear functional F on l^2 such that $F(x) = 0$ for $x \in l^1$. Furthermore, since $(l^2)^2 = l^1$, F is zero on $(l^2)^2$. Now we define, for $x \in A$, $x = (x_1, x_2)$,

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$$q(x) = \left| \sum_{i=1}^{\infty} x_1(i) + F(x_2) \right|.$$

We note the following properties of $q(x)$:

(1) $$q(x+y) \leqq q(x) + q(y),$$

$$\begin{aligned} q(xy) &= \left| \sum_{i=1}^{\infty} x_1(i)y_1(i) + F(x_2y_2) \right| \\ &= \left| \sum_{i=1}^{\infty} x_1(i)y_1(i) \right| \\ &\leqq \left(\sum_{i=1}^{\infty} |x_1(i)|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} |y_1(i)|^2 \right)^{1/2} \\ &\leqq p(x)p(y). \end{aligned} \tag{2}$$

Finally, we define $\|x\| = \max(p(x), q(x))$.

LEMMA 1. $\|\cdot\|$ *is a norm on* A.

PROOF. We verify only that $\|xy\| \leqq \|x\| \, \|y\|$. $\|xy\| = \max(p(xy), q(xy))$. But $p(xy) \leqq p(x)p(y)$ and $q(xy) \leqq p(x)p(y)$ by (2) above, and, therefore, $\|xy\| \leqq p(x)p(y) \leqq \|x\| \, \|y\|$.

Let e_k and f_k be the elements of A defined by

$$\begin{aligned} (e_k)_1(i) &= \delta_{ik}, \quad (e_k)_2(i) = 0, \quad i \geqq 1, \\ (f_k)_1(i) &= 0, \quad (f_k)_2(i) = \delta_{ik}, \quad i \geqq 1. \end{aligned}$$

The set of all e_k and f_k is the set of minimal idempotents of A.

LEMMA 2. *A has dense socle w.r.t. (with respect to)* $\|\cdot\|$.

PROOF. Assume that $x \in A$ and $F(x_2) = \lambda$. Define y_n, an element of the socle of A, by

$$y_n = \left(\sum_{i=1}^{n} x_1(i)e_i + \sum_{i=1}^{n} \frac{\lambda}{n} e_i, \ \sum_{j=1}^{n} x_2(j)f_j \right).$$

Then $p(x-y_n)^2 = \sum_{i=n+1}^{\infty} |x_1(i)|^2 + |\lambda|^2/n + \sum_{j=n+1}^{\infty} |x_2(j)|^2$, and, therefore, $p(x-y_n) \to 0$ as $n \to \infty$.

$$\begin{aligned} q(x-y_n) &= \left| \sum_{i=n+1}^{\infty} x_1(i) - \sum_{i=1}^{n} \frac{\lambda}{n} + \lambda \right| \\ &= \left| \sum_{i=n+1}^{\infty} x_1(i) \right|, \end{aligned}$$

and so, also, $q(x-y_n)\to 0$ as $n\to\infty$.

Hence $\|x-y_n\|\to 0$ as $n\to\infty$, and it follows that the socle of A is dense in A.

LEMMA 3. *A_2 is a closed ideal w.r.t.* $\|\cdot\|$, *and A_2 does not have dense socle w.r.t.* $\|\cdot\|$.

PROOF. A_2 is clearly an annihilator ideal of A, and, therefore, A_2 is closed in any norm topology on A.

Now choose $x_2\in A_2$ such that $F(x_2)=1$. Note that, if y_2 is an element of the socle of A_2, $F(y_2)=0$. Therefore, $\|(0, x_2)-(0, y_2)\| \geqq q((0, x_2-y_2))=|F(x_2)|=1$.

Since y_2 was an arbitrary element of the socle of A_2, A_2 does not have dense socle.

At this point we list results from the theory of annihilator and dual algebras which we use in what follows:

If B is a semi-simple commutative normed algebra in which every maximal modular ideal is closed, then

(i) If B has dense socle, B is an annihilator algebra [**3**, Theorem (2.8.29), p. 106].

(ii) If B is an annihilator algebra, B has dense socle [**3**, Corollary (2.8.16), p. 100].

(iii) If B is dual, every closed ideal of B is dual [**3**, Theorem (2.8.14), p. 100].

The theorems referred to in (i)–(iii) are stated for the case where B is a Banach algebra, but the proofs hold in the more general case where every maximal modular ideal of B is closed.

LEMMA 4. *Every maximal modular ideal of A is closed w.r.t.* $\|\cdot\|$.

PROOF. A is a Banach algebra in the norm $|x|=\sum_{i=1}^{\infty}|x_1(i)| +(\sum_{i=1}^{\infty}|x_2(i)|^2)^{1/2}$. By [**3**, Theorem (2.8.29), p. 106], A is dual w.r.t. $|\cdot|$. In particular, every maximal modular ideal of A is an annihilator ideal, and hence closed w.r.t. any norm topology.

THEOREM. (1) *A is an annihilator algebra w.r.t.* $\|\cdot\|$.

(2) *A_2 is not an annihilator algebra w.r.t.* $\|\cdot\|$.

(3) *A is not dual w.r.t.* $\|\cdot\|$.

PROOF. By Lemma 4, (i)–(iii) apply to A. By (i) and Lemma 2, A is an annihilator algebra w.r.t. $\|\cdot\|$.

It is easily seen that every maximal modular ideal of A_2 is closed w.r.t. $\|\cdot\|$. Therefore (ii) applies, and (ii) and Lemma 3 imply that A_2 is not an annihilator algebra w.r.t. $\|\cdot\|$.

Finally if A were dual w.r.t. $\|\cdot\|$, A_2 would be dual by (iii). This contradicts part (2).

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