ON THE FACIAL STRUCTURE OF CONVEX POLYTOPES[1]

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A finite family **C** of convex polytopes in a Euclidean space shall be called a *complex* provided

(i) every face of a member of **C** is itself a member of **C**;

(ii) the intersection of any two members of **C** is a face of both.

If P is a d-polytope (i.e., a d-dimensional convex polytope) we shall denote by $\mathbf{B}(P)$ the *boundary complex* of P, i.e., the complex consisting of all faces of P having dimension $d-1$ or less. By $\mathbf{C}(P)$ we shall denote the complex consisting of all the faces of P; thus $\mathbf{C}(P)=\mathbf{B}(P)\cup\{P\}$. For a complex $\mathbf{C}$ we define $\operatorname{set}(\mathbf{C})=\bigcup_{C\in\mathbf{C}} C$. For an element C of a complex $\mathbf{C}$ the *closed star* [*anti-star*] of C (in $\mathbf{C}$) is the smallest subcomplex of $\mathbf{C}$ containing all the members of $\mathbf{C}$ which contain C [do not meet C]. The *linked complex* of C in $\mathbf{C}$ is the intersection of the closed star of C with the anti-star of C.

A complex $\mathbf{C}$ is a *refinement* of a complex $\mathbf{K}$ provided there exists a homeomorphism ϕ carrying $\operatorname{set}(\mathbf{C})$ onto $\operatorname{set}(\mathbf{K})$ such that for every $K\in\mathbf{K}$ there exists a subcomplex $\mathbf{C}_K$ of $\mathbf{C}$ with $\phi^{-1}(K)=\operatorname{set}(\mathbf{C}_K)$.

For example, the complex $\mathbf{K}_1$ consisting of two triangles with a common edge is a refinement of the complex $\mathbf{K}_2$ consisting of one triangle; note, however, that the 1-skeleton of $\mathbf{K}_1$ is *not* a refinement of the 1-skeleton of $\mathbf{K}_2$. Let Δ^d denote the d-simplex. The following result is simple but rather useful:

THEOREM 1. *For every d-polytope P the complex* $\mathbf{C}(P)$ *is a refinement of* $\mathbf{C}(\Delta^d)$.

PROOF. The assertion of the theorem is obviously equivalent to the following statement:

THEOREM 1*. *For every d-polytope P the complex* $\mathbf{B}(P)$ *is a refinement of* $\mathbf{B}(\Delta^d)$.

We shall prove the theorem in the second formulation, using induction on d. The case $d=1$ being trivial, we may assume $d\geqq 2$. Let V be a vertex of P and let H be a $(d-1)$-plane intersecting (in relatively interior points) all the edges of P incident to V. Then $P_0=P\cap H$ is a $(d-1)$-polytope, and, by the inductive assumption, $\mathbf{B}(P_0)$ is a refinement of $\mathbf{B}(\Delta^{d-1})$. Let $\mathbf{S}$ denote the closed star of V in $\mathbf{B}(P)$.

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Using radial maps from V, it is obvious that the linked complex $\boldsymbol{L}$ of V in $\boldsymbol{B}(P)$ (i.e., the subcomplex of $\boldsymbol{S}$ consisting of all the members of $\boldsymbol{S}$ which do not contain V) is a refinement of $\boldsymbol{B}(P_0)$ and thus of $\boldsymbol{B}(\Delta^{d-1})$, while $\boldsymbol{S}$ is a refinement of the closed star $\boldsymbol{S}^*$ of a vertex of $\boldsymbol{B}(\Delta^d)$.

On the other hand, denoting by $\boldsymbol{A}$ the anti-star of V in $\boldsymbol{B}(P)$, set($\boldsymbol{A}$) is homeomorphic to the $(d-1)$-cell Δ^{d-1} by a homeomorphism carrying set($\boldsymbol{L}$) onto the boundary of Δ^{d-1}. Since $\boldsymbol{L}$ is a refinement of $\boldsymbol{B}(\Delta^{d-1})$, it follows that $\boldsymbol{A}$ is a refinement of $\boldsymbol{C}(\Delta^{d-1})$. Together with the earlier established fact that $\boldsymbol{S}$ is a refinement of $\boldsymbol{S}^*$ this implies (since, on $\boldsymbol{L}$, the two refinements may be chosen to coincide) that $\boldsymbol{B}(P)$ is a refinement of $\boldsymbol{B}(\Delta^d)$, as claimed.

As an immediate consequence of Theorem 1 we obtain the following result [2, Theorem 3]:

COROLLARY 1. *Every d-polyhedral graph contains a refinement of the complete graph with $d+1$ nodes.*

REMARK. The author is indebted to Dr. Micha Perles for the observation that the proof of Corollary 1, as given in [2], is incomplete, and for indicating how the construction in [2] has to be changed in order to yield a satisfactory proof.

Theorem 1 yields trivially also the following generalization of Corollary 1:

COROLLARY 2. *For every k, $0 \leqq k \leqq d$, the k-skeleton of any d-polytope contains a refinement of the k-skeleton of Δ^d.*

We recall the interesting result of Flores [1] (see also Hurewicz-Wallman [3, p. 63]):

The n-skeleton of Δ^{2n+2} is not homeomorphic to a subset of Euclidean $2n$-space.

Using Schlegel-diagrams, Corollary 2 and Flores' theorem imply:

THEOREM 2. *The n-skeleton of a $(2n+1)$-polytope is not homeomorphic to the n-skeleton of a d-polytope for $d \geqq 2n+2$.*

REFERENCES

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