"RECURSIVE ISOMORPHISM" AND EFFECTIVELY EXTENSIBLE THEORIES[1]

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This paper is concerned with the following problem: to what extent can we obtain a meaningful classification of mathematically interesting formal theories by virtue of their recursive properties? Myhill's results on the "recursive isomorphism" of creative theories [2] indicate that it may not be sufficient to identify a theory with its set of theorems. To what extent can a "recursive isomorphism" preserving the deductive structure of the theories provide a meaningful classification? Accordingly we will concern ourselves with how a theory is presented in terms of axioms and rules of inference (see 2 below).

DEFINITIONS. 1. A *theory* $\mathfrak{T}_i$ is an ordered triple $\langle W_i, T_i, R_i \rangle$, where W_i is a recursive set and T_i and R_i are recursively enumerable sets satisfying $T_i \subseteq W_i$ and $R_i \subseteq W_i$. Theory $\mathfrak{T}_i$ is *consistent* if $T_i \cap R_i = \emptyset$.

Intuitively, W_i stands for the set (of Gödel numbers) of all statements, T_i the set of theorems and R_i the set of refutable statements. Thus $W_i - (T_i \cup R_i)$ represents the set of undecidable statements. Note that all theories considered in this paper are axiomatizable. They are also assumed to be consistent.

2. We now define a *presentation*. If $\mathfrak{T}$ possesses negation the following definition suffices. A *presentation of a theory* is an ordered pair $\langle \alpha, \boldsymbol{R} \rangle$, α a recursively enumerable set (the set of axioms) and $\boldsymbol{R}$ a recursively enumerable sequence of recursively enumerable relations (the rules of inference). Furthermore, $\phi \in T$ if and only if ϕ can be obtained from a finite number of members of α by a finite number of applications of finitely many members of $\boldsymbol{R}$. Note that membership in R is determined by membership in T.

If $\mathfrak{T}$ does not possess negation the above definition must be modified so that the presentation possesses a component which generates

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[2] Some of these results were presented to the Association of Symbolic Logic, April 21, 1964.

the refutables. One way of doing this is as follows. A presentation is an ordered triple $\langle \alpha, \boldsymbol{R}, \nu \rangle$, where α and $\boldsymbol{R}$ are as above, ν is a recursively enumerable relation and $R = \{\nu(\phi) \mid \phi \in T\}$. Since we are assuming all our theories possess negation—see last paragraph of this introduction—the first definition will suffice for our needs.

For technical reasons we assume our presentations $\langle \alpha, \boldsymbol{R} \rangle$ of $\mathfrak{J}$ satisfy the following property. Let $\omega_i \subseteq W$. If ω_i is consistent with respect to $\langle \alpha, \boldsymbol{R} \rangle$ and if $\phi \in W - [T_{\langle \alpha \cup \omega_i, \boldsymbol{R} \rangle} \cup R_{\langle \alpha \cup \omega_i, \boldsymbol{R} \rangle}]$, then ϕ is consistent with $\langle \alpha \cup \omega_i, \boldsymbol{R} \rangle$.

3. A theory is *effectively extensible* if there exists a presentation $\langle \alpha, \boldsymbol{R} \rangle$ and a recursive function f associated with this presentation such that, if ω_i is a recursively enumerable set of sentences which is consistent with $\langle \alpha, \boldsymbol{R} \rangle$ (in the sense that the presentation $\langle \alpha \cup \omega_i, \boldsymbol{R} \rangle$ determines a consistent theory), $f(i)$ is an undecidable sentence of the extension $\langle \alpha \cup \omega_i, \boldsymbol{R} \rangle$.[3]

The concept of effective extensibility provides an abstract formalization for the following strong form of Gödel's theorem. Given a particular presentation of number theory, one can exhibit an effective procedure which, when applied to a consistent axiomatizable extension of the theory, gives an undecidable statement of this extension. Note that distinct presentations yield distinct effective procedures since the effective method depends on the notion of a proof (and, hence, on the axioms and rules of inference).

We assume that all our theories possess negation. For most of our theorems it is possible to obtain analogous theorems for the case in which the theories do not possess negation.[4]

Summary of results.

A. *Negation-preserving recursive maps.* The results in this section represent a sampling of the corollaries of two theorems. The theorems themselves are too complicated to state in this brief account.

I. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be two effectively extensible theories. There exists a 1-1 recursive function mapping W_1 *onto* W_2, T_1 *onto* T_2, R_1 *onto* R_2, $W_1 - (T_1 \cup R_1)$ *onto* $W_2 - (T_2 \cup R_2)$ and *preserving negation.*

II. Let $\mathfrak{J}_1$ be a consistent theory; let $\mathfrak{J}_2$ be an effectively extensible theory. There exists a 1-1 recursive function mapping W_1 *into* W_2,

[3] Some argument can be given for augmenting the rules of inference of the theory effectively. We shall not do that in this paper. In actual practice—for example, in the work *Undecidable theories* [4]—extensions of a theory usually are obtained by including additional formulas as axioms while keeping the rules of inference the same.

[4] *Added in proof, April 2, 1965.* By *negation* we mean a unary connective satisfying the following: (1) $\phi \in R \leftrightarrow \text{neg } \phi \in T$, (2) $\phi \in T \leftrightarrow \text{neg } \phi \in R$; *neg* is a 1-1 recursive function mapping W into W.

T_1 *into* T_2, R_1 *into* R_2, $W_1-(T_1\cup R_1)$ *into* $W_2-(T_2\cup R_2)$ and *preserving negation.*

For theories with a presentation in standard formalization, the negation-preserving recursive maps of I and II can be chosen to preserve the deductive structure of the theories to the extent that first steps of a deduction in $\mathfrak{J}_1$ are mapped onto first steps of a deduction in $\mathfrak{J}_2$. More precisely,

III. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be two effectively extensible theories with presentations in standard formalization. There exists a 1-1 recursive function mapping W_1 *onto* W_2, T_1 *onto* T_2, R_1 *onto* R_2, $W_1-(T_1\cup R_1)$ *onto* $W_2-(T_2\cup R_2)$ and *preserving negation* such that

(1) tautologies of $\mathfrak{J}_1$ are mapped onto tautologies of $\mathfrak{J}_2$,

(2) sentences with tautologous negations are mapped onto sentences with tautologous negations,

(3) logical axioms are mapped onto logical axioms.

Furthermore, if none of the nonlogical axioms is of the form $\neg\psi$, where ψ is a sentence and if, in addition, the cardinal number of both sets of nonlogical axioms is the same, then

(4) nonlogical axioms are mapped onto nonlogical axioms.

IV. Let $\mathfrak{J}_1$ be a consistent theory; let $\mathfrak{J}_2$ be an effectively extensible theory. Suppose both theories have presentations in standard formalization. Then there is a 1-1 recursive function mapping W_1 *into* W_2, T_1 *into* T_2, R_1 *into* R_2, $W_1-(T_1\cup R_1)$ *into* $W_2-(T_2\cup R_2)$ and *preserving negation* such that (1), (2), (3), (4) of III hold.

Theorems I–IV provide generalizations of results in [1], [2], [3]. Note that the exact analogy of Myhill's theorem fails: there exist two creative theories $\mathfrak{J}_1$ and $\mathfrak{J}_2$ such that no 1-1 recursive function mapping W_1 onto W_2 and T_1 onto T_2 also preserves negation.

Since many interesting undecidable theories are effectively extensible, they are "recursively isomorphic" in the sense of I or III by a negation-preserving recursive mapping. Thus these generalizations are as objectionable as Myhill's theorem itself as a basis for classifying mathematically interesting formal theories. Now the above mappings do not take into account much of the deductive structure of the formal theories. For most mathematically interesting formal theories, the deductive structure is intimately related to the deduction theorem and modus ponens. Hence, the preservation of the deductive structure is closely connected with the preservation of implication. We show, in contrast to the foregoing results, that

B. *Nonpreservation of implication.*

V. There exist two effectively extensible theories $\mathfrak{J}_1$ and $\mathfrak{J}_2$, each possessing a presentation with modus ponens as the sole rule of inference, such that no 1-1 recursive function mapping W_1 onto W_2 and T_1 onto T_2 preserves both implication and negation.

C. *Effective extensibility in theories which possess some arithmetic.*

VI. Suppose $\mathfrak{J}$ is consistent and has a presentation as an applied predicate calculus with identity which contains both a notation for the natural numbers and a formula $x \leqq y$ satisfying the following:

1. For each natural number n, $\vdash_{\mathfrak{J}}(x)\ (x \leqq \bar{n} \rightarrow x = \bar{0} \vee \cdots \vee x = \bar{n})$.
2. For each natural number n, $\vdash_{\mathfrak{J}}(x)\ (x \leqq \bar{n} \vee \bar{n} \leqq x)$.

Suppose that every *primitive* recursive function of one argument is definable in $\mathfrak{J}$.

Then there is a formula $\Phi(x)$, with one free variable, such that, *if* ω_i is a recursively enumerable set (of Gödel numbers) of sentences consistent with $\mathfrak{J}$, *then* the formula $\Phi(\bar{\imath})$—with Gödel number $\phi(i)$—is an undecidable sentence of the extension obtained by adding as axioms all sentences Ψ such that the Gödel number of Ψ belongs to ω_i.

It is obvious that all theories which satisfy the hypothesis of VI are effectively extensible. These theories include any consistent extension of the theory R of *undecidable theories*. Thus, any consistent extension $\mathfrak{J}$ of R has the property that undecidable statements of consistent extensions of $\mathfrak{J}$ may be chosen as substitution instances of one specific formula of the theory.

D. *Relation between effective extensibility, and effective inseparability.*

VII. If a theory is effectively extensible with respect to one presentation, it is effectively extensible with respect to all.

VIII. A theory is effectively extensible if and only if it is effectively inseparable (see [**3**] for definition).

IX. The function f which witnesses the fact that a theory is effectively extensible may be chosen to be of the form $f(x) = gt(x)$, where g is a recursive permutation and t is a primitive recursive function.

The proofs of the two theorems of which I–IV are corollaries represent an extension and generalization of the method of [**2**]. The proof of VI proceeds by constructing a suitable Rosser-like statement. Results VII–IX are obtained from reducibility considerations. A detailed account of the proofs of these theorems together with additional consequences is planned for a later publication.

The referee has called our attention to two papers which seem to be related to some of the above material, especially to Theorem VIII [Uspenskij, *Theorem of Gödel and theory of algorithms*, see review, J. Symbolic Logic **19**, 218, and Ehrenfeucht, Bull. Polon. Acad. Sci. **9**, **17**]. Certain aspects of these concepts remain to be investigated.

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REPORT ON ATTAINABILITY OF SYSTEMS OF IDENTITIES

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1. **Introduction.** This note is to report the main results in the paper, *Attainability of systems of identities on semigroups*, which will be published elsewhere with detailed proof.

Let f and g be words, i.e., finite sequences of letters. By an identity we mean an equality $f=g$ of two words f and g. Let $\mathfrak{J}$ be a system of identities T_λ,

$$\mathfrak{J} = \{T_\lambda; \lambda \in \Lambda\} \quad \text{where } T_\lambda \text{ is "} f_\lambda = g_\lambda, \text{"}$$

for example, $\{xyz=xzy,\ x=x^2\}$, $\{xy=yx,\ x=x^2\}$ and so on.

Let S be a semigroup. For a fixed S and a fixed $\mathfrak{J}$, consider the set $\mathfrak{C}$ of all congruences ρ on S such that S/ρ satisfies $\mathfrak{J}$, in other words, $\mathfrak{J}$ identically holds if all letters are replaced by elements of S/ρ. There is the smallest element ρ_0 in $\mathfrak{C}$ in the sense that $\rho_0 \subseteq \rho$ for all $\rho \in \mathfrak{C}$ [1], [4], [7], [8], [9], [11]. Then ρ_0 is called the smallest $\mathfrak{J}$-congruence, and the partition of S due to ρ_0 is called the greatest $\mathfrak{J}$-decomposition. Of course, such a decomposition of S is unique. If the cardinal number $|S/\rho_0|$ of S/ρ_0 is greater than 1, then S is called $\mathfrak{J}$-decomposable; if $|S/\rho_0|=1$, then S is $\mathfrak{J}$-indecomposable. In particular, if $\mathfrak{J}$ is a semilattice, that is, $\mathfrak{J}=\{x=x^2,\ xy=yx\}$, then ρ_0 is called the smallest semilattice-congruence or, simply, s-congruence. The author proved in his papers [8], [10] the following theorem, and also Petrich recently proved the equivalent statement [6].