FUNCTIONS WITH THE HUYGENS PROPERTY

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A C^2 function $u(x, t)$ belongs to class H, for $a<t<b$, and is called a generalized temperature function, if and only if it is a solution of the generalized heat equation

$$\Delta_x u(x, t) = (\partial/\partial t)\, u(x, t),$$

where $\Delta_x f(x) = f''(x) + (2\nu/x)f'(x)$, ν a fixed positive number. The fundamental solution of this equation is

$$G(x, y; t) = (1/2t)^{\nu+1/2}\mathcal{G}(xy/2t) \exp[-(x^2 + y^2)/4t],$$

with $\mathcal{G}(z) = c_\nu z^{1/2-\nu} I_{\nu-1/2}(z)$, $c_\nu = 2^{\nu-1/2}\Gamma(\nu+\frac{1}{2})$, and $I_\gamma(z)$ the Bessel function of order γ of imaginary argument. We write $G(x; t)$ for $G(x, 0; t)$. The function $u(x, t)$ is said to have the Huygens property, that is, it belongs to class H^*, for $a<t<b$, if and only if $u(x, t) \in H$ there, and

$$u(x, t) = \int_0^\infty G(x, y; t - t')u(y, t')\, d\mu(y), \qquad d\mu(x) = (1/c_\nu)x^{2\nu}\, dx,$$

for every t, t', $a<t'<t<b$, the integral converging absolutely. A generalized heat polynomial $P_{n,\nu}(x, t)$ is defined by

$$P_{n,\nu}(x, t) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} [\Gamma(\nu + \tfrac{1}{2} + n)/\Gamma(\nu + \tfrac{1}{2} + n - k)]x^{2n-2k}t^k,$$

and its Appell transform $W_{n,\nu}(x, t)$ is given by

$$W_{n,\nu}(x, t) = G(x, t)P_{n,\nu}(x/t, -1/t).$$

The object of this paper is to summarize the principal results derived in characterizing a generalized temperature function which may be represented either by the series expansion $\sum_{n=0}^\infty a_n P_{n,\nu}(x, t)$ or by $\sum_{n=0}^\infty b_n W_{n,\nu}(x, t)$, with convergence taken in the L^2, as well as in the pointwise, sense. Details and proofs will appear later. The work is an extension of the theory developed by Rosenbloom and Widder in [3]. Some of the preliminary results for this study were also de-

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rived by F. M. Cholewinski, and it has been called to our attention that Louis Bragg has done work in this area.

The region of convergence of the series $\sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t)$ is, in general, a strip $|t| < \sigma$, whereas that of the series $\sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t)$ is a half plane $t > \sigma \geqq 0$. Indeed, we have

THEOREM 1. *If* $\lim_{n\to\infty} |a_n|^{1/n} 4n/e = 1/\sigma < \infty$, *then the series*

$$\sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t)$$

converges absolutely in the strip $|t| < \sigma$ *and does not converge everywhere in any including strip.*

THEOREM 2. *If* $\lim_{n\to\infty} |b_n|^{1/n} 4n/e = \sigma < \infty$, *then the series*

$$\sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t)$$

converges absolutely in the half plane $t > \sigma \geqq 0$ *and does not converge everywhere in any including half plane.*

Within their regions of convergence, $\sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t)$ and $\sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t)$ each defines a generalized temperature function $u(x, t)$, with the additional fact, in the first case, that $u(x, 0)$ is an even entire function of growth $(1, 1/4\sigma)$. An entire function $\phi(x) = \sum_{n=0}^{\infty} c_n x^n$ is said to be of growth (ρ, τ) if and only if $\lim_{n\to\infty} n|c_n|^{\rho/n} \leqq \tau e\rho$.

Since we find that $u(x, t)$ has an expansion $\sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t)$ in the largest strip $|t| < \sigma$ for which $u(x, t) \in H^*$, we note that the role of membership in class H^* in expansions in terms of generalized heat polynomials is analogous to that of analyticity in expansions in Taylor series. This is established in the following.

THEOREM 3. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t),$$

the series converging for $|t| < \sigma$, *is that* $u(x, t) \in H^*$ *there. The coefficients* a_n *have either of the determinations*

$$a_n = u^{(2n)}(0, 0)/(2n)!, \quad or$$

$$a_n = \left\{\Gamma(\nu + \tfrac{1}{2})/[2^{4n} n! \Gamma(\nu + \tfrac{1}{2} + n)]\right\} \int_0^{\infty} u(y, -t) W_{n,\nu}(y, t)\, d\mu(y), \quad 0 < t < \sigma.$$

In addition, the following result gives us a complex determination of the coefficients.

THEOREM 4. *If* $u(x, t)=\sum_{n=0}^{\infty} a_n P_{n,\nu}(x, t)$, *the series converging for* $|t|<\sigma$, *then*

$$a_n = \left\{(-1)^n\Gamma(\nu+\tfrac{1}{2})/[2^{4n}n!\Gamma(\nu+\tfrac{1}{2}+n)]\right\}\int_0^\infty u(ix, t)W_{n,\nu}(x, t)\,d\mu(x),$$
$$0<t<\sigma.$$

Membership in class H^* is not sufficient for an expansion in terms of $W_{n,\nu}(x, t)$, as is indicated by the function $u(x, t)=1$, which is in H^* for $-\infty<x<\infty$ but cannot be represented by the expansion $\sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t)$. Instead we have the following modification of the dual to Theorem 3.

THEOREM 5. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t),$$

the series converging for $t>\sigma\geqq 0$, *is that* $u(x, t)\in H^*$ *there and that*

$$\int_0^\infty |u(x, t)|\, e^{x^2/8t}\,d\mu(x) < \infty, \qquad \sigma<t<\infty.$$

The coefficients b_n *have the determination*

$$b_n = \left\{\Gamma(\nu+\tfrac{1}{2})/[2^{4n}n!\Gamma(\nu+\tfrac{1}{2}+n)]\right\}\int_0^\infty u(y, t)P_{n,\nu}(y, -t)\,d\mu(y),$$
$$\sigma<t<\infty.$$

The proof of this theorem depends on one which establishes necessary and sufficient conditions of a different nature for such an expansion.

THEOREM 6. *A necessary and sufficient condition that*

$$u(x, t) = \sum_{n=0}^{\infty} b_n W_{n,\nu}(x, t),$$

the series converging for $t>\sigma\geqq 0$, *is that*

$$u(x, t) = \int_0^\infty \mathcal{J}(xy)e^{-ty^2}\phi(y)\,d\mu(y), \qquad \sigma<t<\infty,$$

where $\mathcal{J}(z)=2^{\nu-1/2}\Gamma(\nu+1/2)z^{1/2-\nu}J_{\nu-1/2}(z)$, $\phi(y)$ *is an even entire function of growth* $(1, \sigma)$, *and* $b_n=\phi^{(2n)}(0)/[(2n)!(-4)^n]$.

For expansions in the L^2 sense, we have the following:

THEOREM 7. *If* $u(x, t)\in H^*$, $-\sigma\leqq t<0$, *and if* $u(x; t)[G(x; -t)]^{1/2}\in L^2$, *for each fixed* t, $-\sigma\leqq t<0$, $0\leqq x<\infty$, *then, for* $-\sigma\leqq t<0$,

$$\lim_{N\to\infty}\int_0^\infty G(x, -t)\left|u(x, t)-\sum_{n=0}^N a_nP_{n,\nu}(x, t)\right|^2 d\mu(x) = 0,$$

and

$$\int_0^\infty G(x, -t)\,|\,u(x, t)\,|^2\,d\mu(x) = \sum_{n=0}^\infty 2^{4n}n!\Gamma(\nu+\tfrac{1}{2}+n)\,|\,a_n\,|^2 t^{2n}/\Gamma(\nu+\tfrac{1}{2}),$$

where

$$a_n = \{\Gamma(\nu+\tfrac{1}{2})/[2^{4n}n!\Gamma(\nu+\tfrac{1}{2}+n)]\}\int_0^\infty u(y, t)W_{n,\nu}(y, -t)\,d\mu(y),$$
$$-\sigma\leqq t<0.$$

The example $u(x, t)=e^{a^2t}\mathcal{J}(ax)$ illustrates a limitation of this theorem. Although, in this case, $u(x, t)\in H^*$ for $0<t<\infty$, as well as for $-\infty<t\leqq 0$, so that it may be represented by $\sum_{n=0}^\infty a_nP_{n,\nu}(x, t)$, with convergence in the pointwise sense for $0<t<\infty$, Theorem 7 fails to give such an expansion in the L^2 sense for $0<t<\infty$, since $u(x, t)[G(x, -t)]^{1/2}\notin L^2$ for $0<t<\infty$. Thus, for $t>0$, we need an additional result.

THEOREM 8. *If* $u(x, t)\in H^*$, $0<t\leqq\sigma$, *and if* $u(ix, t)[G(x, t)]^{1/2}\in L^2$, *for each fixed* t, $0<t\leqq\sigma$, $0\leqq x<\infty$, *then, for* $0<t\leqq\sigma$,

$$\lim_{N\to\infty}\int_0^\infty G(x, t)\left|u(ix, t)-\sum_{n=0}^N a_nP_{n,\nu}(x, -t)\right|^2 d\mu(x) = 0,$$

and

$$\int_0^\infty G(x, t)\,|\,u(ix, t)\,|^2\,d\mu(x) = \sum_{n=0}^\infty 2^{4n}n!\Gamma(\nu+\tfrac{1}{2}+n)\,|\,a_n\,|^2 t^{2n}/\Gamma(\nu+\tfrac{1}{2}),$$

where

$$a_n = \{\Gamma(\nu+\tfrac{1}{2})/[2^{4n}n!\Gamma(\nu+\tfrac{1}{2}+n)]\}\int_0^\infty u(ix, t)W_{n,\nu}(x, t)\,d\mu(x),$$
$$0<t\leqq\sigma.$$

The dual to Theorem 7 is the following:

THEOREM 9. *If* $u(x, t) \in H^*$, $t \geqq \sigma > 0$, *and if* $u(x, t)[G(ix, t)]^{1/2} \in L^2$, *for each fixed* t, $t \geqq \sigma > 0$, $0 \leqq x < \infty$, *then for* $t \geqq \sigma > 0$,

$$\lim_{N \to \infty} \int_0^\infty G(ix, t) \left| u(x, t) - \sum_{n=0}^{N} b_n W_{n,\nu}(x, t) \right|^2 d\mu(x) = 0,$$

and

$$\int_0^\infty G(ix, t) \mid u(x, t) \mid^2 d\mu(x)$$
$$= \sum_{n=0}^{\infty} 2^{4n-2\nu-1} n! \Gamma(\nu + \tfrac{1}{2} + n) \mid a_n \mid^2 t^{-2n-2\nu-1} / \Gamma(\nu + \tfrac{1}{2}),$$

where

$$b_n = \left\{ \Gamma(\nu + \tfrac{1}{2}) / [2^{4n} n! \Gamma(\nu + \tfrac{1}{2} + n)] \right\} \int_0^\infty u(x, t) P_{n,\nu}(x, -t) \, d\mu(x),$$
$$0 \leqq t < \infty.$$

The properties of $P_{n,\nu}(x, t)$ and $W_{n,\nu}(x, t)$ play a central role in the development of the theory. Of primary importance is the fact that the polynomials $P_{n,\nu}(x, t)$ and the functions $W_{n,\nu}(x, t)$ form a biorthogonal system in the sense that

$$\int_0^\infty W_{n,\nu}(x, t) P_{m,\nu}(x, -t) \, d\mu(x) = \delta_{mn} 2^{4n} n! \Gamma(\nu + \tfrac{1}{2} + n) / \Gamma(\nu + \tfrac{1}{2}).$$

In addition, the equation

$$G(x, y; s - t) = \sum_{n=0}^{\infty} \left\{ \Gamma(\nu + \tfrac{1}{2}) / [2^{4n} n! \Gamma(\nu + \tfrac{1}{2} + n)] \right\} W_{n,\nu}(y, s) P_{n,\nu}(x, -t)$$

is fundamental. We make repeated use, in the proofs, of asymptotic estimates of $P_{n,\nu}(x, t)$ and $W_{n,\nu}(x, t)$. Indeed, these estimates, in addition to the fact that $P_{n,\nu}(x, t) \in H^*$, for $-\infty < t < \infty$, and $W_{n,\nu}(x, t) \in H^*$, for $0 < t < \infty$, enable us to prove that the integral determinations of the coefficients in the above series expansions are all independent of t.

REFERENCES

1. F. M. Cholewinski and D. T. Haimo, *The Weierstrass-Hankel convolution transform*, J. Analyse Math. (to appear).

2. D. T. Haimo, *Generalized temperature functions*, Duke Math. J. (to appear).

3. P. C. Rosenbloom and D. V. Widder, *Expansions in terms of heat polynomials and associated functions*, Trans. Amer. Math. Soc. 92 (1959), 220–266.

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