TOPOLOGY OF QUATERNIONIC MANIFOLDS

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We give here a quaternionic analogue (Theorem 4) of the Hodge decomposition theorem [2, p. 26] for a Riemannian manifold with holonomy group contained in $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$. Applying Chern's theorem in [1] (also [3]), we obtain some consequences on Betti numbers (Theorem 5).

Let $\boldsymbol{K}^n$ denote the n-dimensional vector space over the field $\boldsymbol{K}$ of quaternions, with the inner product

$$(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n} (p_i\bar{q}_i + q_i\bar{p}_i),$$

where

$$\boldsymbol{p} = (p_1, \cdots, p_n), \qquad \boldsymbol{q} = (q_1, \cdots, q_n) \text{ and}$$
$$p_i = \overset{0}{p_i} + \overset{1}{p_i}i + \overset{2}{p_i}j + \overset{3}{p_i}k,$$
$$q_i = \overset{0}{q_i} + \overset{1}{q_i}i + \overset{2}{q_i}j + \overset{3}{q_i}k$$

are quaternions.

Let $\mathrm{Sp}(n)$ be the set of all endomorphisms, A, of $\boldsymbol{K}^n$, satisfying the identity $(A\boldsymbol{p}, A\boldsymbol{q}) = (\boldsymbol{p}, \boldsymbol{q})$. $\mathrm{Sp}(n)$ is the set of all $n\times n$ matrices preserving the inner product. Then $\mathrm{Sp}(1)$ is the set of all unit quaternions. We define the action of $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ on $\boldsymbol{K}^n$ as follows:

$$(A, \lambda)\boldsymbol{p} = A\boldsymbol{p}\lambda, \quad \text{for } (A, \lambda) \in \mathrm{Sp}(n) \times \mathrm{Sp}(1),$$

i.e., we multiply $\boldsymbol{p}$ on the left by the matrix A and on the right by the unit quaternion λ.

DEFINITION. We define three skew symmetric bilinear forms Ω_I, Ω_J and Ω_K on $\boldsymbol{K}^n$ as follows:

$$\Omega_I(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}i, \boldsymbol{q}),$$
$$\Omega_J(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}j, \boldsymbol{q}) \text{ and}$$
$$\Omega_K(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}k, \boldsymbol{q}).$$

Note that Ω_I, Ω_J and Ω_K may be thought of as exterior 2-forms of $\boldsymbol{K}^n$ considered as a $4n$-dimensional real vector space.

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DEFINITION. We define an exterior 4-form Ω on K^n by

$$\Omega = \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K.$$

THEOREM 1. *Ω is invariant under the action of* $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$.

THEOREM 2. $\Omega^n = \Omega\wedge\Omega\wedge \cdots \wedge \Omega$ (*n times*) $\neq 0$.

DEFINITION. A $4n$-dimensional Riemannian manifold M is called a *quaternionic manifold* if its holonomy group is a subgroup of $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$.

If M is a quaternionic manifold of dimension $4n$, then, by Theorems 1 and 2, we have a differential 4-form Ω on M of maximal rank (i.e., $\Omega^n \neq 0$) which is parallel. Hence, Ω is a harmonic form. From the fact that $\Omega^n \neq 0$, we have

THEOREM 3. *If B_i denotes the ith Betti number of a quaternionic manifold M of dimension $4n$, then we have $B_{4i} \neq 0$ for $i = 0, 1, \cdots, n$.*

We define the operator * which sends a p-form into a $(4n-p)$-form in the usual way.

DEFINITION. Define two operators L and Λ on the differential forms by

$$Lw = \Omega \wedge w, \qquad \Lambda w = *(\Omega \wedge *w).$$

A differential form w is called *effective* if $\Lambda w = 0$.

THEOREM 4. *Let w be a p-form; then*

$$w = w_e^{p} + Lw_e^{p-4} + \cdots + L^r w_e^{p-4r}, \quad \textit{for } p \leqq n,$$

where w_e^k is an effective k-form, and $r = [p/4]$.

From Theorem 4, it follows that L sending p-forms into $(p+4)$-forms is 1-1 for $p \leqq n-4$.

THEOREM 5. *We have an increasing sequence of Betti numbers,*

$$B_i \leqq B_{i+4} \leqq \cdots \leqq B_{i+4r}, \quad \textit{for } i + 4r \leqq n,\ i = 0, 1, 2 \textit{ or } 3.$$

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