THE HAUPTVERMUTUNG AND THE POLYHEDRAL SCHOENFLIES THEOREM

BY P. M. RICE

Communicated by M. L. Curtis, December 28, 1964

1. **Introduction.** M. L. Curtis [1] has conjectured that the double suspension of a Poincaré manifold is a 5-sphere. If this is true, it gives counterexamples to the Hauptvermutung, the closed star conjecture, and the polyhedral Schoenflies theorems. We prove here that the only way to get a noncombinatorial triangulation of a manifold is, essentially, to multiply suspend a combinatorial manifold which is not a sphere. As a corollary, we establish that, modulo the Poincaré conjecture, one of the polyhedral Schoenflies theorems is equivalent to the Hauptvermutung.

2. **Terminology.** The Hauptvermutung is the conjecture that any two triangulations of an n-manifold are piecewise linearly homeomorphic. It is convenient to consider two conjectures which together imply the Hauptvermutung. The first is that any triangulation of an n-manifold is combinatorial (meaning that the link of any vertex is a combinatorial $(n-1)$-sphere), and the second is that any two combinatorial triangulations of an n-manifold are piecewise linearly homeomorphic. We will call the first of these $H(n)$. $H(n)$ is known for $n=1, 2, 3$. PS(n) will denote the conjecture that, if a combinatorial $(n-1)$-sphere S is embedded as a subcomplex of a triangulated n-sphere T, then S is locally flat in T. PS(n) is known for $n=1, 2, 3$. $P(n)$ will be the n-dimensional Poincaré conjecture, which is known except for $n=3, 4$. S^n will be any space homeomorphic to the n-sphere, $X \cong Y$ means X is homeomorphic to Y, $X \circ Y$ is the topological join of X and Y, and $S(X)$ is the suspension of X.

3. **Main result.**

THEOREM. *If there is a noncombinatorial triangulation of an n-manifold M, then there is a combinatorial m-manifold K^m, $m \geqq 3$, such that*
(i) *K^m is a homology m-sphere but $K^m \not\cong S^m$ and*
(ii) $K^m \circ S^{n-m-1} \cong S^n$.

PROOF. Let v be a vertex of M such that LK(v, M), the link of v in M, is not a combinatorial $(n-1)$-sphere. If LK$(v, M)=K^{n-1}$ is a combinatorial manifold, then $S(K^{n-1}) \cong S^n$ by Theorem 4 of [2] and the theorem is proved. By induction, if $K^p \circ S^{n-p-1} \cong S^n$ but K^p is not a combinatorial manifold, then, for some vertex w of K^p, $\mathrm{LK}(w, K^p) = K^{p-1}$ is not a combinatorial $(p-1)$-sphere. But $\mathrm{St}(w, K^p) = w \circ K^{p-1}$ and $\mathrm{St}(w, K^p) \circ S^{n-p-1} \cong E^n$, so $(w \circ K^{p-1}) \circ S^{n-p-1} = w \circ (K^{p-1} \circ S^{n-p-1})$ is locally n-euclidean at w, and, by Theorem 4 of [2],

$$S(K^{p-1} \circ S^{n-p-1}) = K^{p-1} \circ S^{n-p} \cong S^n.$$

For some $m \geqq 3$, K^m is a combinatorial manifold, for, if not, then $K^2 \circ S^{n-3} \cong S^n$, so K^2 is a polyhedral homology manifold with the homology of S^2, whence $K^2 \cong S^2$, and, since $H(2)$ is true, K^2 is a combinatorial 2-sphere, from which it would follow that K^3 is a combinatorial manifold.

COROLLARY 1. *Suppose that a combinatorial $(n-1)$-sphere is embedded as a subcomplex of a triangulated n-sphere such that the closure of one complementary domain is a combinatorial n-cell. If this implies that the other complementary domain is simply connected, then $P(3)$ and $P(4)$ imply $H(n)$.*

PROOF. Let K^m be as in the theorem, and v a vertex of K^m. $\mathrm{St}(v, K^m) \circ S^{n-m-1}$ is a combinatorial n-cell embedded as a subcomplex of $K^m \circ S^{n-m-1} \cong S^n$. Then

$$(K^m \circ S^{n-m-1}) \backslash (\mathrm{St}(v, K^m) \circ S^{n-m-1}) = (K^m \backslash \mathrm{St}(v, K^m)) \times E^{n-m}$$

is simply connected. Thus K^m is a simply connected, combinatorial, homology sphere, which, by the Poincaré conjecture, is an m-sphere.

COROLLARY 2. *$P(3)$, $P(4)$, $PS(n)$ implies $H(n)$, and $H(n)$ implies $PS(n)$.*

PROOF. The last statement follows from the combinatorial Schoenflies theorem.

REFERENCES

1. M. L. Curtis and E. C. Zeeman, *On the polyhedral Schoenflies theorem*, Proc. Amer. Math. Soc. **11** (1960), 888–889.

2. R. H. Rosen, *Stellar neighborhoods in polyhedral manifolds*, Proc. Amer. Math. Soc. **14** (1963), 401–406.

UNIVERSITY OF GEORGIA