A NEW INVARIANT OF HOMOTOPY TYPE AND SOME DIVERSE APPLICATIONS

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Let X be a connected, locally finite simplicial polyhedron. Let X^X be the space of maps from X to X with the compact-open topology. Let $x_0 \in X$ be taken as a base point in X, then the evaluation map $p: X^X \to X$ defined by $p(f) = f(x_0)$ for $f \in X^X$ is continuous. Now p induces the homomorphism

$$p_*: \pi_1(X^X, 1_X) \to \pi_1(X, x_0),$$

where $1_X \in X^X$ is the identity map. Hence $p_*\pi_1(X^X, 1_X)$ is a subgroup of the fundamental group of (X, x_0).

PROPOSITION 1. *$p_*\pi_1(X^X, 1_X)$ considered as a subgroup of $\pi_1(X, x_0)$ is an invariant of homotopy type.*

In [2], this invariant is studied and theorems are obtained which bear on the study of X^X, groups of homeomorphisms, homological group theory and knot theory. Most of these results come from the following theorem.

THEOREM 2. *Let X have the homotopy type of a compact, connected polyhedron with nonzero Euler-Poincaré number. Then $p_*\pi_1(X^X, 1_X) = 0$.*

The proof of this employs Nielsen-Wecken fixed-point class theory ([1] and [5]).

Let $G(X)$ be the group of homeomorphisms of a manifold X, and let $G_0(X)$ be the isotropy group over x_0. Then there is an exact sequence [3]

$$\cdots \to \pi_i(G_0(X), 1_X) \xrightarrow{i_*} \pi_i(G(X), 1_X) \xrightarrow{p'_*} \pi_i(X, x_0) \to \cdots,$$

where $p': G(X) \to X$ is the evaluation map.

COROLLARY 3. *Let X be as in Theorem 2. Then $p'_*\pi_1(G(X), 1_X) = 0$. In particular, if $\pi_2(X, x_0) = 0$, then $i_*: \pi_1(G_0(X), 1_X) \cong \pi_1(G(X), 1_X)$.*

This follows because $p'_*\pi_1(G(X), 1_X) \subseteq p_*\pi_1(X^X, 1_X)$.

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THEOREM 4. *If X is an aspherical polyhedron, then $p_*\pi_1(X^X, 1_X) = Z(\pi_1(X, x_0))$, the center of $\pi_1(X, x_0)$.*

Theorems 2 and 4 combine to give us the following corollaries:

COROLLARY 5. *If X has the same homotopy type as a compact, connected, aspherical polyhedron with nonzero Euler-Poincaré number, then $Z(\pi_1(X, x_0)) = 0$.*

John Stallings, in [4], has put this result in a purely algebraic setting; namely, if a group G admits a finite resolution, then, if $Z(G)$ is nontrivial, the (suitably defined) Euler-Poincaré number is zero.

Alexander's Duality and the last corollary gives us a result suggested by L. P. Neuwirth.

COROLLARY 6. *Suppose that X is a subcomplex of the n-sphere S^n whose Euler characteristic is different from that of S^n. If $S^n - X$ is connected and aspherical, then $\pi_1(S^n - X)$ has no center.*

Finally, we are able to show the following:

THEOREM 7. *If X is aspherical, then*

$$\pi_1(X^X, 1_X) \cong Z(\pi_1(X, x_0)),$$
$$\pi_n(X^X, 1_X) \cong 0, \qquad n > 1.$$

Note that Theorem 7 and Theorem 2 give us:

COROLLARY 8. *If X has the homotopy type of an aspherical compact polyhedron whose Euler characteristic is different from zero, then the identity component of X^X is contractible.*

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