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HANKEL TRANSFORMS AND ENTIRE FUNCTIONS

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Communicated by R. Boas, November 9, 1964

Paley and Wiener proved that every entire function of exponential type τ which belongs to L^2 in the real axis can be represented as the Fourier transform of a function which belongs to $L^2(-\tau, \tau)$ and conversely (see Boas [1, p. 103]). The L^p -analogue of the Paley-Wiener theorem for $1 < p < 2$ was proved by Boas [2] and by Plancherel and Pólya [9]. Boas also showed that the theorem does not hold for other values of p unless some restrictions are imposed. The extensions to functions of order $1/m$, where m is an integer ≥ 1 , and type σ are given by Ibragimov [7]. Since the Hankel transforms are natural generalizations of the Fourier transforms, it is natural to ask whether such a representation for entire functions is possible in this case also. The aim of this note is to obtain an analogue of the Paley-Wiener theorem for Hankel transforms for the case $1 < p < 2$ and to extend the results of Ibragimov. These results with proofs will appear elsewhere and we shall only summarize them here.

Unless otherwise stated, ν is always assumed to be greater than or equal to $-1/2$. If $p > 1$, then q will denote its conjugate index given by $p^{-1} + q^{-1} = 1$. Let $z = x + iy$ denote the complex variable. $J_\nu(z)$ denotes the Bessel function of the first kind of order ν .

The Hankel transform of a function $f(x) \in L^p(0, \infty)$, $p > 1$, is defined by the formula

$$F(u) = \int_0^\infty (xu)^{1/2} J_\nu(xu) f(x) dx,$$

where the integral is taken in the L^q -sense or in the mean, that is,

¹ Presented to the Society on April 19, 1963. A major part of this paper forms part of the author's dissertation at Northwestern University (August, 1963), written under the direction of Professor R. P. Boas, Jr. This research was partially supported by NSF Grant GP-314.

$$\lim_{a \rightarrow \infty} \int_0^a \left| F(u) - \int_0^a (xu)^{1/2} J_\nu(xu) f(x) dx \right|^q du = 0.$$

The existence of $F(u)$ for $1 < p \leq 2$ is contained in a theorem of Titchmarsh [11].

The L^2 -analogue of the Paley-Wiener theorem for Hankel transforms was proved by Griffith [5, pp. 109-115].

Our principal result is

THEOREM I. *Let $f(z)$ be an even entire function of exponential type 1. If $1 < p \leq 2$ and $x^{p+1/2}f(x) \in L^p(-\infty, \infty)$, then $f(z)$ can be represented by*

$$(1) \quad f(z) = z^{-\nu} \int_0^1 t^{-\nu} J_\nu(zt) \phi(t) dt,$$

where $t^{-\nu-1/2}\phi(t) \in L^q(0, 1)$. If $f(z)$ has the representation (1) and $t^{-\nu-1/2}\phi(t) \in L^p(0, 1)$, $1 < p \leq 2$, then $f(z)$ is an even entire function of exponential type 1 such that $x^{p+1/2}f(x) \in L^q(-\infty, \infty)$.

We shall point out that the example given by Boas [2] for the Fourier case can be suitably modified to show that, if $p > 2$, there exists an even entire function $f(z)$ of exponential type 1 such that $x^{p+1/2}f(x) \in L^p(-\infty, \infty)$ but not of the form (1) with $t^{-\nu-1/2}\phi(t) \in L^q(0, 1)$.

The proof of the second part of Theorem I is fairly easy while that of the first part depends on Theorem II which is also of independent interest.

THEOREM II. *A necessary and sufficient condition that $f(z)$ has the form (1) with $t^{-\nu-1/2}\phi(t) \in L^p(0, 1)$, $p > 1$, is that the following hold:*

(a) $f(z)$ is an even entire function of exponential type 1,

$$(b) \quad f(j_n) = j_n^{-\nu} \int_0^1 t^{-\nu} J_\nu(j_n t) \phi(t) dt,$$

(c) $x_n^{p+1/2}f(x_n) \rightarrow 0$ as $n \rightarrow \infty$, where x_n are points on the real axis such that

$$\liminf |x_n - j_n| > 0$$

and

$$|x_n - j_n| < \pi/2,$$

where j_n is the n th positive zero of $J_\nu(z)$.

Theorem II is established by techniques analogous to those of Boas [2].

From Theorem I we obtain, by appropriate changes of variable, representations for functions of order $1/m$ and type σ such that either

$$\int_0^{\infty} |x^{\alpha} f(x)|^p x^{-(m-1)/m} dx \quad \text{or} \quad \int_0^{\infty} |x^{\alpha} f(x)|^p x^{(p-1)(m-1)/m} dx$$

is finite, where $\alpha \geq 0$ and m is an integer ≥ 1 . These include as particular cases (for $\nu = \pm \frac{1}{2}$) all the results given by Ibragimov [7, pp. 63–73].

As applications we get various inequalities, e.g., for $\theta^m f(z)$, where $\theta = (z^{-2\nu-1}D)(z^{2\nu+1}D)$ and $D = d/dz$, and more general operators, and for $|f'(z)|$ and $|f(z)|$ in terms of $\int_{-\infty}^{\infty} |x^{\nu+1/2} f(x)|^p dx$.

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