MORSE THEORY FOR G-MANIFOLDS

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Morse theory relates the topology of a Hilbert manifold [3, §9], M, to the behavior of a C^∞ function f: $M \rightarrow \boldsymbol{R}$ having only nondegenerate critical points. In applying Morse theory to the study of G-manifolds, i.e., manifolds with a compact Lie group G acting as a differentiable transformation group, one must, of course, use maps in the category, i.e., equivariant maps. However, if x is a critical point of an equivariant function then gx is also a critical point for any $g \in G$, hence one must allow critical orbits or, more generally, critical submanifolds.

In §1 we give the necessary definitions and notation. In §2 we extend the results of R. Palais in [3] to study an invariant C^∞ function f: $M \rightarrow \boldsymbol{R}$ on a complete Riemannian G-space M, where in addition to f satisfying condition (C) [3, §10], we require that the critical locus of f be a union of nondegenerate critical manifolds in the sense of Bott [1]. In §3 we show that if M is finite-dimensional then any invariant C^∞ function on M can be C^k approximated by a C^∞ invariant function whose critical orbits are nondegenerate. Together with the results of §2 this provides an analogue for G-manifolds of the Smale handlebody decomposition technique. Proofs will be given elsewhere.

1. **Notation and definition.** G will denote a compact Lie group and M a C^∞ Hilbert manifold. If ψ: $G \times M \rightarrow M$ is the differentiable action of G on M, then, for any $g \in G$, $\bar{g}$: $M \rightarrow M$ will denote the map given by $\bar{g}(m) = \psi(g, m)$; $\psi(g, m)$ will also be shortened to gm. If M, N are G-manifolds, then f: $M \rightarrow N$, is equivariant if $f \circ \bar{g} = \bar{g} \circ f$ for all $g \in G$; f is invariant if $f \circ \bar{g} = f$ for all $g \in G$. The tangent bundle $T(M)$ of a G-manifold M is a G-manifold with the action $gX = d\bar{g}_p(X)$, for $X \in T(M)_p$. If E and B are G-manifolds and π: $E \rightarrow B$ is a Hilbert vector bundle [2], then π is said to be a G-vector bundle if, for each $g \in G$, $\bar{g}$: $E \rightarrow E$ is a bundle map. Note that π is then equivariant as is the zero-section. If, in addition, π has a Riemannian metric, $\langle\ ,\ \rangle$, and each $g \in G$ acts isometrically, then π will be called a Riemannian G-vector bundle. M will be called a Riemannian G-space if $T(M) \rightarrow M$ is a Riemannian G-vector bundle. Let f: $M \rightarrow \boldsymbol{R}$ be an invariant C^∞ function. The gradient vector field, ∇f, on M, is defined by $\langle \nabla f, X \rangle = df_p(X)$ for $X \in T(M)_p$ and, since f is invariant, $g\nabla f_p$, $\langle X \rangle = \langle \nabla f_p, g^{-1}X \rangle = df_p(g^{-1}X) = d(f \circ \bar{g}^{-1})_{gp}(X) = df_{gp}(X) = \langle \nabla f_{gp}, X \rangle$ for all $X \in T(M)_{gp}$

so $g\nabla f_p = \nabla f_{gp}$. Hence, if σ_p is the maximum solution curve of ∇f with initial condition p [3, §6], then $g\sigma_p = \sigma_{gp}$.

At a critical point of p, i.e., where $\nabla f_p = 0$, we have a bounded, self-adjoint operator, the hessian operator, $\phi(f)_p = T(M)_p \to T(M)_p$, defined by $\langle \phi(f)_p v, w\rangle = H(f)_p(v, w)$, where $H(f)_p$ is the hessian bilinear form [3, §7]. A closed invariant submanifold V of M will be called a *critical manifold* of f if $\partial V = \varnothing$, $V \cap \partial M = \varnothing$ and if each $p \in V$ is a critical point of f. It follows that $T(V)_p \subseteq \ker \phi(f)_p$, and so there is an induced bounded self-adjoint operator $\bar{\phi}(f)_p$: $T(M)_p/T(V)_p \to T(M)_p/T(V)_p$. If $\bar{\phi}(f)_p$ is an isomorphism for each $p \in V$, then V is called a *nondegenerate critical manifold of f*.

Recall that f is said to satisfy condition (C) if each subset S of M on which f is bounded but on which $\|\nabla f\|$ is not bounded away from zero has a critical point of f in its closure.

DEFINITION. The invariant C^∞ function of f: $M \to \mathbf{R}$ is called a *Morse function* for the Riemannian G-manifold M if it satisfies condition (C) and if the critical locus of f is a union of nondegenerate critical manifolds without interior.

If E is a Riemannian G-vector bundle or Hilbert space then $\|e\| = \langle e, e\rangle^{1/2}$ and $E(r) = \{e \in E \mid \|e\| \leqq r\}$, $E^\circ(r) = \{e \in E \mid \|e\| < r\}$ and $\dot{E}(r) = \{e \in E \mid \|e\| = r\}$. If f: $M \to \mathbf{R}$, then $f^{a,b}$ will denote $\{m \in M \mid a \leqq f(m) \leqq b\}$ and $f^b = f^{-\infty,b}$.

$C_G(M)$ will denote the invariant C^∞ functions on the finite-dimensional G-manifold M with the C^k topology for some fixed $k \geqq 2$. If $f \in C_G(M)$, $\epsilon > 0$ and ψ: $\mathbf{R}^n \to M$ is a coordinate chart for M, then a neighborhood of f in the C^k topology is given by

$$\{h \in C_G(M) \mid N_k(f \circ \psi - h \circ \psi)(x) < \epsilon \text{ for } \|x\| \leqq 1\},$$

where

$$N_k(f \circ \psi)(x) = \sum_{j=0}^{k} \|d^j(f \circ \psi)_x\|,$$

and $\|\ \|$ denotes the usual norm on multilinear transformations. $C_G(M)$ is a space of the second category.

2. **Morse functions.** The behavior of a function near a critical manifold is specified by the

MORSE LEMMA. *Let π: $E \to B$ be a Riemannian G-vector bundle and f a Morse function on E having B (i.e., the zero-section) as a nondegenerate critical manifold. If B is compact there is an equivariant diffeomorphism θ: $E(r) \to E$ for some $r > 0$ such that $f(\theta(e)) = \|Pe\|^2 - \|(1-P)e\|^2$, where P is an orthogonal bundle projection.*

An important property of Morse functions is given by:

PROPOSITION. *If f is a Morse function the critical locus of f in $f^{a,b}$ is the union of a finite number of disjoint, compact, nondegenerate critical manifolds of f.*

We also have the

DIFFEOMORPHISM THEOREM. *Let f be a Morse function on M with no critical value in the bounded interval $[a, b]$. If $f^{a-\delta,b+\delta}$ is complete for some $\delta > 0$ then f^a is equivariantly diffeomorphic to f^b.*

Attaching a handle-bundle.

DEFINITION. Let V, W be Riemannian G-vector bundles over B. The bundle $V(1) \oplus W(1) = \{(x, y) \in V \oplus W \mid \|x\| \leqq 1, \|y\| \leqq 1\}$ (not a manifold) is called a handle-bundle of type (V, W) with index = dimension of W. Let N, M be manifolds with boundary, $N \subset M$ and f: $V(1) \oplus W(1) \rightarrow M$ a homeomorphism onto a closed subset H of M. We shall write $M = N \cup_f H$, and say that M arises from N, by attaching a handle-bundle of type (V, W) if

(i) $M = N \cup H$,

(ii) $f \mid \dot{V}(1) \oplus W(1)$ is a diffeomorphism onto $H \cap \partial N$,

(iii) $f \mid V^\circ(1) \oplus W(1)$ is a diffeomorphism onto $M - N$.

ATTACHING LEMMA. *Let π: $E \rightarrow B$ be a Riemannian G-vector bundle and P an orthogonal bundle projection. Let $V = P(E)$, $W = (1-P)(E)$ and define f, g: $E \rightarrow \boldsymbol{R}$ by $f(e) = \|Pe\|^2 - \|(1-P)e\|^2$, $g(e) = f(e) - 3\epsilon/2\lambda(\|Pe\|^2/\epsilon)$ where $\epsilon > 0$ and λ is the function defined in [3, §11]. Then $\{x \in E(2\epsilon) \mid g(x) \leqq -\epsilon\}$ arises from $\{x \in E(2\epsilon) \mid f(x) \leqq -\epsilon\}$ by attaching a handle-bundle of type (V, W).*

Note that B is a nondegenerate critical manifold of f. By the Morse Lemma we can choose coordinates for π: $E \rightarrow B$ such that $f(e) = \|Pe\|^2 - \|(1-P)e\|^2$ in a neighborhood of B for any function f having B as a nondegenerate critical manifold. Hence, by abuse of notation, we shall also refer to the handle-bundle of type $(P(E), (1-P)E)$ as the handle-bundle (B, f).

MAIN THEOREM. *Let f be a Morse function on the complete Riemannian G-space M. If f has a single critical value c in the bounded interval $[a, b]$, then the critical locus of f in $[a, b]$ is the disjoint union of a finite number of compact submanifolds $N_1, \cdots, N_s$. f^b is equivariantly diffeomorphic to f^a with s handle-bundles of type (N_i, f) disjointly attached.*

An excision and Thom's theorem proves the

COROLLARY (BOTT [1]). *Let $N_1, \cdots, N_t$ be those critical manifolds with index $(N_i, f) = k_i < \infty$. Then*

$$H_m(f^b, f^a; Z_2) \approx \sum_{i=1}^{t} H_{m-k_i}(N_i; Z_2).$$

Now let a, b be arbitrary regular values of f, $a<b$, and again denote the critical manifolds of finite index k_i by $\{N_i\}$, $i=1, \cdots, t$. Let $R_m(X)$ = dimension of $H_m(X; Z_2)$ and $\chi(X)$ the Euler characteristic of X. Then we have the Morse inequalities:

(i) $\chi(f^b, f^a) = \sum_{i=1}^{t} (-1)^{k_i}\chi(N_i)$,

(ii) $R_m(f^b, f^a) \leqq \sum_{i=1}^{t} R_{m-k_i}(N_i)$,

(iii) $\sum_{l=0}^{m} (-1)^{m-l}R_l(f^b, f^a) \leqq \sum_{i=1}^{t}\sum_{l=0}^{m} (-1)^{m-l}R_{l-k_i}(N_i)$.

3. **Density lemma.** Let M be a finite-dimensional G-manifold. For any compact subset A of M, $\mathfrak{M}_G(A, M) \subset C_G(M)$ will denote those functions whose critical locus in A is the union of nondegenerate critical orbits. Clearly $\mathfrak{M}_G(A, M)$ is open in $C_G(M)$.

LEMMA 1. *Let G act orthogonally on the Euclidean space V with fixed point set W. Then $\mathfrak{M}_G(W(1), V)$ is open and dense in $C_G(V)$.*

The proof follows from an application of Sard's theorem to $f|W$ (for any f) and some jiggling of f in the normal direction to W. Baire's theorem and a double induction on the dimension and number of components of M yields

LEMMA 2. *$\mathfrak{M}_G(V(1), V)$ is open and dense in $C_G(M)$.*

One further application of Baire's theorem yields

DENSITY LEMMA. *For any finite-dimensional G-manifold M, $\mathfrak{M}_G(M, M)$ is dense in $C_G(M)$.*

Carefully approximating an invariant proper function by a function in $\mathfrak{M}_G(M, M)$ gives

COROLLARY. *There exists a Morse function on M.*

Combining the corollary with the main theorem yields

COROLLARY. *If M is compact then $M = (N_1, f) \cup_{g_1} (N_2, f) \cdots \cup_{g_s} (N_s, f)$ where the (N_j, f)'s are handle-bundles over orbits.*

Vector bundles over orbits can be described as follows: Let $\pi: E \rightarrow \Omega$ be a G-vector bundle over the orbit Ω, $x \in \Omega$ and let $H \subset G$ be the isotropy group of x. Then $\Omega \approx G/H$ and $G \rightarrow G/H$ is a principal bundle with structural group H. Since H acts linearly on $\pi^{-1}(x) = F$ we have the associated vector bundle $G \times_H F$ with fibre F. $G \times_H F \rightarrow G/H$ is actually a G-vector bundle since the actions of G and H on $G\times F$ commute. The projection $G\times F\to F$ extends by equivariances to a bundle equivalence

$$\begin{array}{ccc} G\times_H F & \to & E \\ \downarrow & & \downarrow \\ G/H & \approx & \Omega. \end{array}$$

Hence $\pi\colon E\to\Omega$ is determined by the action of H on F.

References

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SYMPLECTIC GROUPS OVER DISCRETE VALUATION RINGS

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A symplectic group over a field $\neq F_2$ or F_3, according to a theorem of Dickson and Dieudonné (see [1]), has no normal subgroups other than its center $\{\pm 1\}$. Attempts at integral analogues of this theorem have of late been quite successful. First Klingenberg [6] showed that every normal subgroup of a symplectic group over a local ring is a congruence group (again with some exceptions). Then Bass, Lazard and Serre [2] showed that every normal subgroup of finite index in the symplectic group $\mathrm{Sp}_{2n}(Z)$ over the rational integers contains a congruence subgroup if $n\geqq 2$. In [5], Jehne proved local results similar to Klingenberg's, and used them to show that any normal subgroup G of the symplectic group over a suitable Dedekind ring is a congruence subgroup, if G is closed under the congruence topology.

The above three integral results all assumed that the discriminant of the alternating form is a unit. The purpose of this note is to drop this restriction and give a generalization of [6]. In order to obtain a tractable canonical form, it is necessary to assume that the local

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