Lectures on differential geometry. By S. Sternberg. Prentice-Hall, Englewood Cliffs, N. J., 1964. 15+390 pp. $16.00.

At last we have a book that begins with the differentiable manifold concept, develops the tangent vector—vector field—differential form formalism, then proceeds to treat some interesting differential-geometric material in a way that is suitable for use in a graduate course. As a whole, the book lies somewhere between an exposition of topics that interest the author and a systematic treatise. Sternberg's style is vivid and fresh, and, at its high points, the book is an excellent job. However, it has certain defects: Of necessity, most of the space in this review will be spent pointing them out to the reader or teacher of a course based on the book.

Chapter 1 presents the material from multilinear algebra that is needed, following Bourbaki's treatment. The second chapter introduces the fundamental ideas of differentiable manifold, tangent vector, differential form, etc. An unusual feature is that Sard's theorem and various approximation and imbedding theorems are proved at this early stage, so as to introduce non-trivial theorems as soon as possible. This is purposely done before the introduction of the tangent space, but the reviewer can see little point to this. For example, the definition of transversal submanifolds is very awkward in terms of coordinate systems, but would be much simpler if the tangent space were available at this stage. Also, all types of tensor fields are treated in as unified manner as possible (too concisely, in the reviewer's opinion). Most proofs are given using local coordinate systems; while this is not incorrect, it is more consistent with the manifold point of view to do at least part of the work intrinsically, and to use the fact that tensor fields are modules over the ring of functions on the manifold to define the various differential operations on tensor folds. (For example, this is the approach used in Helgason's *Differential geometry and symmetric spaces*, which has in the beginning a treatment of manifolds that requires about the same level of mathematical ability as Sternberg's, and that the reviewer considers more comprehensible.) The standard names associated with the various types of submanifolds have been permuted. For example, a Lie subgroup would be (in the language of this chapter) not a "submanifold," which is standard, but a "one-one immersion." The reviewer would recommend Auslander and Mackenzie's book *Introduction to differentiable manifolds* for an introductory treatment of manifold theory before the beginner tackles this one.

The next chapter is "Integral Calculus on Manifolds." First, the properties of exterior differentiation of differential forms are given, again using local coordinate systems as a shortcut. Then, a very good treatment of forms over simplices and manifolds is presented, with the needed material on integration in Euclidean space sketched in an appendix. One finds an excellent treatment of the degree of a mapping, and various conditions that maps be onto. (Applications of these conditions are given in Chapter 6.) Then comes the theorem on the existence of maximal integral submanifolds for completely integrable differential systems, which is one of the most important in the foundations of Lie groups and differential geometry. The two hardest lemmas in the subject are left to the reader as exercises, with no indications that they are hard. The non-standard names for the various types of submanifolds that were given in the previous chapter are completely ignored here, so that the reader must switch the terminology for himself to make the theorems come out right. The remaining two sections in the chapter are devoted to topics in the geometry of differential forms and to a short discussion of applications to classical mechanics.

The next chapter is on the calculus of variations and Riemannian geometry. Here the author tries to cover a subject which has great depth, complexity and ramifications, in sixty-five pages, which makes it impossible to do an adequate job. First, the emphasis (as is quite right in a book on differential geometry) is on the calculus of variations as a "geometric object." The reader should be somewhat familiar with the more classical approach, e.g. that to be found in an excellent book by Gel′fand and Fomin. A second decision which saves space, but which the reviewer must criticize, is that the Riemannian case is mixed up with the ordinary variational problems given by a general Lagrangian. Here, Sternberg's treatment is traditional, but has been superseded by work due to Ambrose, Berger and Klingenberg (available in lecture note form) who have shown that the Riemannian case is best handled completely independently of the general case. It turns out that the affine connection associated with the metric, if used skillfully and systematically in conjunction with the first and second variation formulas, leads to remarkable simplifications and new insights. However, in Sternberg's treatment affine, connections do not appear at all in this chapter.

The main part of the chapter is a treatment of the conditions for extremals of one-independent variable variational problems, based on the ideas of Cartan and Carathéodory. The discussion is restricted to regular, nonhomogeneous Lagrangians. For example, to treat the Riemannian case it is necessary to minimize the "energy" instead of arc-length. This is justified ("the principle of Maupertuis") but is dependent on the quadratic nature of the Lagrangian; in general, there are no similar simple tricks available for reducing the study of homogeneous Lagrangians to non-homogeneous ones. It is traditional in expositions of the calculus of variations to treat both cases. It would have been wiser to have followed this procedure, since some problems can be treated most naturally with one type rather than the other. For example, consider the ramifications for physics: For non-homogeneous Lagrangians the parametrization of the extremals is not arbitrary, which leads to Newtonian ideas of "time," while the homogeneous Lagrangians, for which the parameter of extremals can be chosen arbitrarily, are appropriate for relativistic mechanics. This distinction becomes even more important in multiple integral variational problems. Sternberg also does not treat the Lagrange variational problems (which are now very important as a source of ideas for optimal control theory).

Now for the details of the chapter. The first problem is to show that every Lagrangian induces a one-differential form on the tangent bundle of the manifold, which the reviewer likes to call the "Cartan-one form." This is done by considering the Legendre transformation as a map from the tangent bundle to the cotangent bundle, then working on the cotangent bundle. In Cartan's original treatment everything was done on the tangent bundle, and the reviewer feels that throwing in the cotangent bundle leads to unnecessary additional complication in an already complicated subject. (Also, working on the tangent bundle only is far simpler when one does the Lagrange variational problem, since the additional constraints imposed there must be regarded as determining a submanifold of the tangent bundle.) In §2, the necessary conditions are carefully worked out. §3 is titled "Conservation Laws." The only conservation law presented here is that of "energy," with the associated Liouville theorem. Unfortunately, this simple case does not illustrate sufficiently well the role that conservation laws play in the calculus of variations and physics. §4 is "Sufficient Conditions." Here Sternberg gives a variant of Carathéodory's method. However, it is extremely short, which will disappoint the reader interested in applications, since Carathéodory's ideas have become very important in optimal control theory. Such important notions as the "extremal vector fields," "Hilbert's independent integral," and the "Hamilton-Jacobi equation" appear, at least implicitly, at this stage, but Sternberg is very careless and brief. The next section, "Conjugate and Focal Points, Jacobi's Condition," is satisfactory, leading to a proof that extremals locally minimize up to the first conjugate point but not beyond, but is not very readable because coordinate systems must be hurriedly introduced. The reviewer would recommend that the reader compare the treatment in the first chapter of Morse's book *The calculus of variations in the large*, or the treatment of the simple case of one dependent variable in Gel'fand and Fomin. On the other hand, it is possible to prove these facts very elegantly by coordinate-free methods for the Riemannian case, so again Sternberg's pedagogical judgment must be criticized. The next two sections, devoted to the Riemannian case, essentially give proof of the Hopf-Rinow theorem concerning completeness of a Riemannian metric. The final section called "Isometries" consists of just one page, and presents one theorem on the fixed point set of isometries that has little relation to the rest of the book. Finally, Sternberg mentions in the preface that "the influence of the late Professor Wintner can be seen in the strong tendency toward classical mechanics." Indeed, it would have been very desirable to see classical mechanics in this chapter, but there is none beyond mention of the "energy" several times and a short treatment of Hamiltonian dynamics in the previous chapter. All in all, this chapter is a severe disappointment to the reader who expects a rather complete modern treatment of the ideas in either Riemannian geometry or the Carathéodory-Cartan approach to the calculus of variations.

Chapter 5 is "Lie Groups." Most of the material is standard and in general done well, with one glaring fault. The most important theorem in Lie group theory is the correspondence between connected Lie subgroups and subalgebras of the Lie algebra. This theorem is not stated, but it is proved casually in the text, while lemmas that are used in the proof are theorems. The beginner will have difficulty, since the subtleties are not clearly explained. A minor point: In the previous chapter, the author mentions explicitly that the reader is not supposed to know Haar measure. On page 234, he uses Haar measure without even giving it that name.

Chapter 6 is "Differential Geometry of Euclidean Space." In essence, this chapter is devoted to the proof of several important global theorems concerning hypersurfaces in Euclidean spaces, associated with such names as Hadamard, Whitney, Chern and Lashof, Hartman and Nirenberg. (This is not quite what Sternberg says he is going to do in the introduction to the chapter, but this can be forgiven since the chapter is very good indeed.) Chapter 7, "The Geometry of G-structures," gives an introduction to the work in differential geometry that Sternberg himself has been doing, and amounts to an attempt to fit the ideas of E. Cartan on general geometric structures and "infinite Lie groups" into the mainstream of modern mathematics. Again, Sternberg's heart has clearly been in this work, and these two chapters are a much better job than the first five. Since most of the material in both of these chapters has not appeared elsewhere in book form, readers primarily interested in differential geometry will find them the most useful and interesting part of the book.

Now for the details of Chapter 7. The first section, "Principal and Associated Bundles, Connections," covers an amazing amount of ground concisely but very well. Here Sternberg gathers together the general ideas that will be used in studying G-structures, which are sub-bundles of the tangent frame bundle. §2 poses the equivalence problem, and defines the set of first order invariants, the structure functions. In certain cases, the structure functions decide the equivalence problem. However, usually they must be used to define a "canonical" reduction of structure group (i.e., one preserved under G-structure isomorphism). Sternberg sketches how this reduction process proceeds. Although it is a sketch, it is the first treatment that the reviewer knows that seems to be a plausible explanation of the process Cartan used in many examples. As application, Sternberg sketches an explanation in reasonable mathematical language of some of the material in Cartan's notoriously difficult paper *Sur les systèmes de Pfaff à cinq variables.* The next section, titled "Prolongations," presents the method that is buried in Cartan's papers on "Infinite Lie Groups" for deciding the equivalence of two G-structures by constructing towers of bundles over the original G-bundle. The most interesting qualitative point here is that one can decide if the prolongation process will end (roughly, whether the group of automorphisms of the original G-structure is a "finite" Lie group) by means of an algebraic property of the group G of $n \times n$ real matrices defining the G-structure. §4 studies these "finite type" G-structures in more detail. The closing sections 5 and 6 fill out the picture with a treatment of linear connections and their geodesics. In this chapter Sternberg gives us, with great expository skill and taste, a glimpse of the vast research areas open to students of differential geometry today in making sense of Cartan's mathematical legacy.

It would be much more useful to the beginner if there were more exercises and examples devoted to non-trivial applications of the generalities to concrete geometric situations, rather than mainly posing as exercises details of proofs that the author does not feel like providing. This applies particularly to the last chapter: The beauty of Cartan's theory of G-structures is that it unifies so much of classical differential geometry. It is pointed out that V. Guillemin has treated in his Harvard thesis the classical geometries (e.g. projective and conformal) as an application of the formalism presented in this chapter. However, this manuscript does not seem to be generally available. It would be helpful if at least this material could be made available as an appendix in a later edition of the book.

In summary, this is a book that contains much useful material and that is, in general, well written, but that is marred by inattention to detail. It can serve well as a graduate textbook on the geometry of manifolds if it is used with care.

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