VARIATIONAL METHODS FOR NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

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In the present note, we give a simple general proof for the existence of solutions of the following two types of variational problems:

PROBLEM A. *To minimize* $\int_\Omega F(x, u, \cdots, D^m u)dx$ *over a subspace* V *of* $W^{m,p}(\Omega)$.

PROBLEM B. *To minimize* $\int_\Omega F(x, u, \cdots, D^m u)dx$ *for* u *in* V *with* $\int_\Omega G(x, u, \cdots, D^{m-1}u)dx = c$.

The solution of the first problem yields a weak solution of a corresponding elliptic boundary-value problem for the Euler-Lagrange equation

$$(1) \qquad Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha F_{p\alpha}(x, u, \cdots, D^m u) = 0.$$

From the solution of the second problem, we obtain a solution under corresponding boundary conditions of the nonlinear eigenvalue problem.

$$(2) \quad Au = \lambda \left\{ \sum_{|\beta| \leq m-1} (-1)^{|\beta|} G_{p\beta}(x, u, \cdots, D^{m-1}u) \right\} = \lambda Bu, \quad \lambda \in R^1.$$

In §1, we give a complete self-contained treatment of the existence of minima of functionals on reflexive Banach spaces, a treatment which extends and strengthens earlier studies by Lusternik, E. Rothe, Vainberg, and others (see [6], [11], [12], [14], [15]). In §2, we apply the results of §1 to Problems A and B, above. In the case of Problem A, we strengthen and simplify results of Morrey [10] and Smale [13]. The relation of the resulting existence theorem for the solution of the variational boundary-value problem for equation (1) to those obtained by the writer in [2], [3], [4] by operator methods (as well as unpublished results of Leray and Lions) and the results of Višik [16] using other analytical methods, is discussed in detail in [5]. Special cases of the eigenvalue problem treated in Problem B have been treated for A linear by Levinson [7] with $A = \Delta$ on R^2, and by Berger [1] for general linear A.

1. Abstract variational problems. Let V be a real Banach space. Strong convergence in V is denoted by $\rightarrow$, weak convergence by $\rightharpoonup$. We consider two functions

$$\Phi\colon V \times V \to R^1,$$
$$g\colon V \to R^1,$$

and define $f\colon V \to R^1$ by $f(v) = \Phi(v, v)$, $v \in V$.

The function Φ is said to be *semi-convex* if both of the following conditions hold:

(a) *For each v in V and c in R^1, the subset $W_{c,v}$ of V given by*

$$W_{c,v} = \{u \mid u \in V,\ \Phi(u, v) \leqq c\} \tag{3}$$

is convex.

(b) *For each bounded set B of V and each sequence $\{v_j\}$ in V with $v_j \rightharpoonup v$, $\Phi(u, v_j) \to \Phi(u, v)$ uniformly for u in B. For fixed v in V, $\Phi(\cdot, v)$ is continuous in the strong topology of V.*

THEOREM 1. *Let V be a reflexive Banach space, Φ a semi-convex real function on $V \times V$. Let $f(v) = \Phi(v, v)$ for v in V, and let C be a weakly closed bounded subset of V. Then f is bounded from below on C and assumes its minimum on C.*

PROOF OF THEOREM 1. We may choose a sequence $\{u_j\}$ from the bounded weakly closed set C such that

$$f(u_j) = \Phi(u_j, u_j) \to c_0 = \operatorname*{g.l.b.}_{u \in C} f(u),$$

while

$$u_j \rightharpoonup u_0, \qquad u_0 \in C.$$

By property (b) of semi-convexity and the boundedness of C,

$$\Phi(u_j, u_j) - \Phi(u_0, u_j) \to 0.$$

Hence

$$\Phi(u_0, u_j) \to c_0.$$

Let c be any real number with $c > c_0$. Then for $j \geqq j_c$, $u_j \in W_{c,u_0}$ as defined by equation (3) above. W_{c,u_0} is convex by property (a), and is closed by the second part of property (b) of semi-convexity. Hence W_{c,u_0} is weakly closed. Since $u_j \rightharpoonup u_0$, $u_0 \in W_{c,u_0}$, i.e., $\Phi(u_0, u_0) \leqq c$. Since c was any number $> c_0$, it follows that $c_0 > -\infty$ and $f(u_0) = c_0$. Q.E.D.

As corollaries of Theorem 1, we have the following:

THEOREM 2. *Let Φ be a semi-convex real function on $V \times V$, where V is a reflexive B-space, and for v in V, let $f(v) = \Phi(v, v)$. If $f(v) \to +\infty$ as $\|v\| \to +\infty$, then f assumes a minimum on V.*

PROOF OF THEOREM 2. Set $C=\{v \mid \|v\| \leqq R\}$ for R sufficiently large.

THEOREM 3. *Let V be a reflexive B-space, Φ a real semi-convex function on $V\times V$, and $f(v)=\Phi(v, v)$ for v in V. Let g be a weakly continuous real function on V. Let $C=\{u \mid g(u)=c\}$ for a fixed c in R^1 and suppose that $f(u)\rightarrow+\infty$ as $\|u\|\rightarrow+\infty$ on C. Then f assumes a minimum on C.*

PROOF OF THEOREM 3. $C\cap\{u \mid \|u\| \leqq R\}$ is weakly closed and bounded for all $R>0$.

Let V^* be the adjoint space of V, (w, u) the pairing between w in V^* and u in V.

If g: $V\rightarrow R^1$, g is said to be differentiable at v_0 in V if there exists an element $g'(v_0)$ in V^* such that for all h in V

$$g(v_0+h) = g(v_0) + (g'(v_0), h) + \epsilon(h)$$

where $\epsilon(h)=o(\|h\|)$ as $\|h\|\rightarrow 0$. If Φ: $V\times V\rightarrow R^1$, Φ is differentiable at (v_1, v_2) if there exists a pair w_1, w_2 in V^* such that

$$\Phi(v_1+h_1, v_2+h_2) = \Phi(v_1, v_2) + (w_1, h_1) + (w_2, h_2) + o(\|h_1\| + \|h_2\|),$$

and we set $w_1=\Phi_1'(v_1, v_2)$, $w_2=\Phi_2'(v_1, v_2)$. If Φ is differentiable at (v_0, v_0) and $f(v)=\Phi(v, v)$, then f is differentiable at v_0 and

$$f'(v_0) = \Phi_1'(v_0, v_0) + \Phi_2'(v_0, v_0).$$

THEOREM 4. *Let V be a Banach space, f and g two real functions on V with f and g differentiable at v_0, $g'(v_0)\neq 0$. If f has a local minimum at v_0 with respect to the set $C=\{v \mid g(v)=g(v_0)\}$, then there exists λ in R^1 such that $f'(v_0)=\lambda g'(v_0)$.*

PROOF OF THEOREM 4. Let $V_1=\{v \mid v\in V,\ (g'(v_0), v)=0\}$, and choose u_0 in V such that $(g'(v_0), u_0)=1$. If v is any element of V_1 with $\|v\|=1$ and ϵ and r are real numbers with $|\epsilon|$, $|r|$ sufficiently small, then

$$f(v_0) \leqq f(v_0 + \epsilon v + r u_0)$$

provided that $g(v_0)=g(v_0+\epsilon_v+ru_0)$. We know that

$$\begin{aligned} g(v_0+\epsilon v+ru_0) &= g(v_0) + \epsilon(g'(v_0), v) + r(g'(v_0), u_0) + s(\epsilon, r, v) \\ &= g(v_0) + r + s(\epsilon, r, v), \end{aligned}$$

where for each fixed v in V_1, $s(\epsilon, r, v)=o(|\epsilon|+|r|)$. Consider r on the interval $[-\frac{1}{2}|\epsilon|, +\frac{1}{2}|\epsilon|]$, and the quantity $r+s(\epsilon, r, v)$ with $\epsilon\neq 0$ and v fixed. For $|\epsilon|$ sufficiently small, $r+s(\epsilon, r, v)$ is negative at the left endpoint, positive at the right, and continuous in r. We may choose a value of $r(\epsilon, v)$ in the interval to make $r+s(\epsilon, r, v)=0$, and hence $|r(\epsilon, v)|=o(|\epsilon|)$. For this choice of r, we have

$$\begin{aligned} f(v_0) &\leqq f(v_0+\epsilon v+ru_0) \\ &= f(v_0)+\epsilon(f'(v_0), v)+r(f'(v_0), u_0)+o(|\epsilon|+|r|) \end{aligned}$$

so that

$$\epsilon(f'(v_0), v) \geqq -o(|\epsilon|), \qquad |\epsilon| \to 0.$$

Hence $(f'(v_0), v)=0$ for all v in V_1 and $f'(v_0)=\lambda g'(v_0)$, for some λ in R^1.

REMARK. In [6] and [14], Theorem 4 is called Lusternik's principle and proofs are given for special cases.

THEOREM 5. *Let V be a reflexive Banach space, Φ a semi-convex real function on $V\times V$, g a weakly continuous real function on V, $f(v)=\Phi(v, v)$ for v in V. Suppose that f and g are differentiable on V, that for a given constant c in R^1 the set $C=\{v \mid g(v)=c\}$ is nonempty, and that $g'(v)\neq 0$ for v in C. Suppose further that $f(v)\to+\infty$ as $\|v\|\to+\infty$ on C, then there exists v_0 in C and λ in R^1 such that $f'(v_0)=\lambda g'(v_0)$.*

Theorem 5 is an immediate consequence of Theorems 3 and 4.

A useful complement to Theorem 5 is the following:

THEOREM 6. *A sufficient condition for condition* (a) *for semi-convexity to hold is that Φ be everywhere differentiable on $V\times V$ and that $\Phi_1'(u, v)$ be monotone in u for fixed v, i.e., for all u_0, u_1 in V,*

$$(\Phi_1'(u_1, v)-\Phi_1'(u_0, v), u_1-u_0) \geqq 0.$$

PROOF OF THEOREM 6. Let u_0, u_1, and v be elements of V, $0\leqq\lambda\leqq 1$. Let $u_\lambda=\lambda u_1+(1-\lambda)u_0$. To prove condition (a), it suffices to show $\Phi(u, v)$ convex in u for fixed v. Set

$$h(\lambda) = \Phi(u_\lambda, v)-\lambda\Phi(u_1, v)-(1-\lambda)\Phi(u_0, v).$$

It suffices to show that $h(\lambda)\leqq 0$ for $0\leqq\lambda\leqq 1$. Since $h(0)=h(1)=0$, it suffices to show that $h'(\lambda)$ is nondecreasing on the interval. However,

$$h'(\lambda) = (\Phi_1'(u_\lambda, v), u_1-u_0)-\Phi(u_1, v)+\Phi(u_0, v),$$

so that for $\lambda<\xi$,

$$\begin{aligned} h'(\xi)-h'(\lambda) &= (\Phi_1'(u_\xi, v)-\Phi_1'(u_\lambda, v), u_1-u_0) \\ &= (\xi-\lambda)^{-1}(\Phi_1'(u_\xi, v)-\Phi_1'(u_\lambda, v), u_\xi-u_\lambda) \geqq 0. \quad \text{Q.E.D.} \end{aligned}$$

REMARK. Connections between monotonicity of the gradient and convexity of the functional have been remarked in Minty [9] and implicitly in Vaĭnberg and Kachurovski [15]. Monotone operators between V and V^* have been studied in Browder [2], [4] and Minty [8].

2. **Nonlinear eigenvalue problems.** We adopt the notation of [2] and [3] in general, except that all our functions will be real- rather than complex-valued. Let Ω be a bounded, smoothly bounded open set in R^n, $n \geqq 1$, D^α the elementary differential operator $(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. We assume that we are given positive integers r and m, a real number p with $1 < p < \infty$, and a closed subspace V of the reflexive Banach space $W^{m,p}(\Omega)$ of r-vector functions u on Ω such that $D^\alpha u \in L^p(\Omega)$ for all α with $|\alpha| \leqq m$. Let $\langle\, , \rangle$ denote the natural inner product in R^r and for two real-valued r-vector functions u and v on Ω, set

$$[u, v] = \int_\Omega \langle u(x), v(x) \rangle \, dx,$$

where the integration is taken with respect to Lebesgue n-measure.

Let $\zeta = \{\zeta_\alpha \mid |\alpha| = m\}$ and $\psi = \{\psi_\xi \mid |\xi| \leqq m-1\}$ be elements of the real vector spaces R^N and R^M, respectively, where for each α and ξ, ζ_α and ψ_ξ are real r-vectors. We assume that we are given two functions

$$F(x, \psi, \zeta), \qquad G(x, \psi)$$

defined on $\Omega \times R^M \times R^N$ and $\Omega \times R^M$, respectively, measurable in x and C^1 in (ψ, ζ) or ψ. We let F_α, F_ξ, and G_ξ denote the appropriate partial gradients of the functions F and G with respect to ζ_α and ψ_ξ.

We suppose that F and G satisfy the following system of inequalities:

$$\text{(I)} \qquad \begin{aligned} |F(x, \psi, \zeta)| &\leqq c(\eta) \Big\{ g(x) + |\zeta|^p + \sum_{|\xi| \leqq m-1} |\psi_\xi|^{p_\xi} \Big\}, \\ |G(x, \psi)| &\leqq c(\eta) \Big\{ g(x) + \sum_{|\xi| \leqq m-1} |\psi_\xi|^{p_\xi} \Big\}, \end{aligned}$$

where $g \in L^p(\Omega)$, p_ξ are exponents satisfying the inequalities

$$(n - p(m - |\xi|))p_\xi < np \quad \text{if } n - p(m - |\xi|) > 0$$

and $c(\eta)$ is a continuous function of $\eta = \{\psi_\xi \mid |\xi| < n/p - m\}$.

For each u in V, let

$$\zeta(u) = \{D^\alpha u \mid |\alpha| = m\}, \quad \psi(u) = \{D^\xi u \mid |\xi| \leqq m - 1\}.$$

Then the functionals

$$\Phi(u, v) = \int_\Omega F(x, \psi(v), \zeta(u))\, dx$$

and

$$g(u) = \int_\Omega G(x, \psi(u))\, dx$$

are well defined and continuous on $V \times V$ and V, respectively, g is weakly continuous on V, and Φ satisfies condition (b) for semi-convexity. If we assume further that:

$$\text{(II)}\qquad \begin{aligned} |F_\alpha(x,\psi,\zeta)| &\leqq c(\eta)\Big\{g_1(x) + |\zeta|^{p-1} + \sum_{|\xi|\leq m-1} |\psi_\xi|^{p'_\xi}\Big\}, \\ |F_\beta(x,\psi,\zeta)| &\leqq c(\eta)\Big\{g_1(x) + |\zeta|^{q_\beta} + \sum_{|\xi|\leq m-1} |\psi_\xi|^{q_{\xi\beta}}\Big\}, \\ |G(x,\psi)| &\leqq c(\eta)\Big\{g_1(x) + \sum_{|\xi|\leq m-1} |\psi_\xi|^{q_{\xi\beta}}\Big\}, \end{aligned}$$

where $g_1 \in L^{p'-1}(\Omega)$ and p'_ξ, q_β, and $q_{\xi\beta}$ are exponents satisfying the inequalities

$$\begin{aligned} (n - p(m - |\xi|))p'_\xi &\leqq n(p-1), \quad \text{if } n - p(m - |\xi|) > 0, \\ (n - p(m - |\xi|))q_{\xi\beta} &\leqq n(p-1) + p(m - |\beta|), \\ nq_\beta &\leqq n(p-1) + p(m - |\beta|) \end{aligned}$$

then the functionals Φ and g are everywhere once differentiable with

$$\begin{aligned} (\Phi'_1(v_1, v_2), u) &= \sum_{|\alpha|=m} [F_\alpha(x, \psi(v_2), \zeta(v_1)), D^\alpha u], \\ (\Phi'_2(v_1, v_2), u) &= \sum_{|\xi|\leq m-1} [F_\xi(x, \psi(v_2), \zeta(v_1)), D^\xi u], \\ (g'(v), u) &= \sum_{|\xi|\leq m-1} [G_\xi(x, \psi(v)), D^\xi u]. \end{aligned}$$

If we assume that each F_α is itself differentiable in ζ and that the following semi-ellipticity condition holds:

$$\text{(III)}\qquad \sum_{|\alpha|,|\beta|=m} \langle F_{\alpha\beta}(x, \psi, \zeta)\eta_\alpha, \eta_\beta\rangle \geqq 0,$$

for all ψ in R^M, x in Ω, and ζ and η in R^N (where $F_{\alpha\beta}$ is the gradient of F_α with respect to ζ_β, which we assume to exist); then Φ will satisfy the monotonicity condition of Theorem 6 and thereby condition (a) for semi-convexity.

Applying Theorem 2, we have:

THEOREM 7. *If F satisfies the inequalities imposed on it in* (I), (II), *and* (III) *and if* $f(v)=\Phi(v, v)\to+\infty$ *as* $\|v\|\to+\infty$, *then* f *has a minimum on V which is a variational solution in the sense of* [2] *of the Euler-Lagrange equation*

$$Au = \sum_{|\beta|\leq m} (-1)^{|\beta|} D^\beta F_\beta(x, \psi(u), \zeta(u)) = 0.$$

Applying Theorem 5, we have:

THEOREM 8. *If F and G satisfy* (I), (II), *and* (III), *if the set* $C=\{v \mid v\in V,\ g(v)=c\}$ *is nonempty and* $g'(v)\neq 0$ *on C, and if, finally,* $f(v)\to+\infty$ *as* $\|v\|\to+\infty$ *on C, then f has a minimum on C which is a variational solution of the appropriate boundary-value problem for the eigenvalue problem*

$$Au = \lambda Bu = \lambda\left\{\sum_{|\xi|\leq m-1} (-1)^{|\xi|} D^\xi G_\xi(x, \psi(u))\right\}.$$

We complete our considerations with the following:

THEOREM 9. (a) *If V is* $W_0^{m,p}(\Omega)$, *the closure of* $C_c^\infty(\Omega)$ *in* $W^{m,p}(\Omega)$, *the boundary conditions in Theorems 7 and 8 are those of the homogeneous Dirichlet problem.*

(b) *Condition* (III) *can be replaced by the weaker integral condition*

$$\sum_{|\alpha|,|\beta|=m} [F_{\alpha\beta}(x, \psi(v_1), \zeta(v_2))D^\alpha u, D^\beta u] \geqq -c\|u\|^p_{m-1,p}.$$

(c) *Theorem 8 can be specialized to hold under the following more intuitive restrictions than* (I) *and* (II): *namely,* $|F| < c\{1+|\zeta|^p+|\psi|^p\}$, $|F_\alpha|+|F_\xi| \leqq c\{1+|\zeta|^{p-1}+|\psi|^{p-1}\}$, $G=u^q$ *with* $q<np(n-pm)^{-1}$ *for* $n>pm$ *and G an arbitrary continuous function of u for* $n<pm$.

The regularity of the solutions of Theorems 7 and 8 can be derived from known results for linear and mildly nonlinear equations A as well as for the case $m=1$, $r=1$.

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