AN EIGENVALUE PROBLEM FOR QUASI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Communicated by F. Browder, September 22, 1964

Eigenvalue problems for nonlinear equations have long been studied in the contexts of abstract function spaces and second-order ordinary differential equations. The present note treats such problems for certain quasi-linear elliptic partial differential equations by means of functional analysis on Sobolev spaces, and extends work in this direction by Levinson [7], Golomb [6], Duff [5], and Vaĭnberg [8]. The variational method used is a direct generalization of the linear case and thus allows the introduction of a simple Hilbert-space approach to this problem.

1. Let G be a fixed bounded domain in real Euclidean N-space R^N with boundary G and closure $\bar{G}=G\cup\partial G$. A general point of G will be denoted $x=(x_1, x_2, \cdots, x_n)$. Integration over G will always be taken with respect to Lebesgue N-dimensional measure. All derivatives are taken in the generalized sense of L. Schwartz. The following notation is very convenient: the elementary differential operators are written

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \qquad (1 \leqq j \leqq N),$$

and for any N-tuple of non-negative integers $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_N)$ the corresponding differential operator of order $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_N$ is written $D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_N^{\alpha_N}$. A linear operator A of order $2m$ is said to be in divergence form if it can be written:

$$Au = \sum_{|\alpha|,|\beta|\leqq m} D^\alpha(a_{\alpha\beta}(x)D^\beta u).$$

If $a_{\alpha\beta}(x)=a_{\beta\alpha}(x)$, A is also formally self-adjoint.

A real linear differential operator A is uniformly elliptic in G if the

[1] This research was partially supported by the National Science Foundation (NSF-GP 2283 and NSF-GP 2280). The results obtained form part of a dissertation submitted at Yale University in candidacy for the degree of Ph.D. The author extends his most hearty thanks to Professor F. E. Browder for constant encouragement and numerous helpful suggestions.

homogeneous characteristic form of A is positive definite, uniformly over G.

$W_{m,p}(G)$ is the collection of functions in $L_p(G)$ for fixed p, $1<p<\infty$, such that $D^\alpha u$, for all $|\alpha|\leqq m$, again lies in $L_p(G)$. $W_{m,p}(G)$ is a Banach space with respect to the norm

$$\|u\|_{m,p} = \left\{ \sum_{|\alpha|\leqq m} \|D^\alpha u\|_{0,p}^p \right\}^{1/p}.$$

In particular, $W_{m,2}(G)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{m,2} = \sum_{|\alpha|\leqq m} \langle D^\alpha u, D^\alpha v\rangle_{0,2}.$$

$\mathring{W}_{m,2}(G)$ is the closure of $C_0^\infty(G)$ in $W_{m,2}(G)$ and thus can be regarded as a Hilbert space.

2. The boundary-value problem to be considered here is:

$$(1) \qquad \begin{aligned} Au - \lambda f(u, x) &= 0, \\ D^\alpha u\big|_{\partial G} &= 0, \qquad 0 \leqq |\alpha| \leqq m-1, \end{aligned}$$

where A is a formally self-adjoint, uniformly elliptic real linear operator of order $2m$ with uniformly bounded, measurable coefficients and top order terms uniformly continuous (A is assumed to be written in divergence form), λ is a real number and $f(t, x)$ is a real-valued function defined on $R^1\times G$, jointly continuous in the x and t variables with the following properties:

1. $f(0, x)=0$.
2. $f(-t, x)=-f(t, x)$.
3. $f(t, x)$ is a nondecreasing function of t for fixed x.
4. For some fixed $x_1\in G$, some positive constant k and all $x\in G$, $f(t, x)\geqq kf(t, x_1)>0$, for $t>0$.
5. (*Polynomial growth condition*) For all $(t, x)\in R^1\times G$, $f(t, x)\leqq k_1+k_2|t|^\rho$, where k_1 and k_2 are non-negative constants and $\rho=\rho(m, N)$. If $f(t, x)$ does not (necessarily) satisfy this condition, we write $\rho=\infty$.

We denote by $Z(\rho_1)$ the family of functions $f(t, x)$ which satisfy the above conditions with

$$\begin{aligned} &0 \leqq \rho < \frac{N+2m}{N-2m} && \text{if } N > 2m, \\ &0 \leqq \rho < \infty && \text{if } N = 2m, \\ &\rho = \infty && \text{if } N < 2m. \end{aligned}$$

DEFINITION 1. A *classical eigenfunction* for the boundary-value problem (1) is a function $u(x)$ with the following properties:

(a) $u(x)$ is $2m$-times continuously differentiable over G.

(b) $u(x)$ is $(m-1)$-times continuously differentiable over $\overline{G}$.

(c) $u(x)$ satisfies the equation $Au=\lambda f(u, x)$ in G.

(d) $u(x)\neq 0$ in G.

(e) $D^\alpha u|_{\partial G}=0$, $0\leqq|\alpha|\leqq m-1$, at each point x of ∂G.

DEFINITION 2. A *generalized eigenfunction* of the boundary-value problem (1) is a function $u(x)$ with the following properties:

(a) $u(x)\in \mathcal{W}_{m,2}(G)$.

(b) $u(x)\neq 0$, apart from a set of measure 0, in G.

(c) $a(u, v)=\lambda\int_G f(u, x)v$ for all $v\in\mathcal{W}_{m,2}(G)$, where $a(u, v)$ is the Dirichlet form associated with the operator A, i.e.,

$$a(u, v) = \sum_{|\alpha|,|\beta|\leq m} \int_G a_{\alpha\beta}(x) D^\alpha u D^\beta v.$$

Differentiation by parts shows that every classical eigenfunction is a generalized eigenfunction. The converse is, in general, not true. (Cf. Theorem III.)

3. THEOREM I. *Suppose $f(t, x)\in Z(\rho_1)$. Then the generalized eigenfunctions of the boundary-value problem* (1) *are identical with the nonzero solutions of the operator equation $\mathfrak{A}u-\lambda Bu=0$ defined on the Hilbert space $\mathcal{W}_{m,2}(G)$; $\mathfrak{A}$ is a self-adjoint bounded linear operator mapping $\mathcal{W}_{m,2}(G)$ into itself and satisfying the inequality*

$$\langle \mathfrak{A}u, u\rangle_{m,2} \geqq c_1\|u\|^2_{m,2} - c_2\|u\|^2_{0,2},$$

where c_1, c_2 are constants with $c_1>0$ and $c_2\geqq 0$; B is a compact, continuous, not necessarily linear mapping of $\mathcal{W}_{m,2}(G)$ into itself with the additional property that $\langle Bu, v\rangle_{m,2}$ is a weakly continuous function of the elements u and v.

PROOF. Define $\langle \mathfrak{A}u, v\rangle_{m,2}=a(u, v)$ and $\langle Bu, v\rangle_{m,2}=\int_G f(u, x)v$. Noticing that both inner products are linear in v, we are able to apply Riesz's representation theorem for linear functionals on the Hilbert space $\mathcal{W}_{m,2}(G)$ to obtain the required operator equation. The inequality satisfied by $\mathfrak{A}$ is a consequence of Gårding's Inequality, while the properties of B follow from Sobolev's Imbedding Theorem.

Set $F(t, x)=\int_0^t f(s, x)ds$.

DEFINITION 3. ∂M_R is the set of all functions $u(x)$ such that:

(a) $u(x)\in\mathcal{W}_{m,2}(G)$,

(b) $\int_G F(u, x)=R$.

(We refer to ∂M_R as the energy level with radius R.)

Lemma 1 (Geometry of Energy Levels). *Let R be a fixed positive number. Then*

(i) ∂M_R *contains elements of* $\mathcal{W}_{m,2}(G)$.

(ii) ∂M_R *is weakly closed and, on* ∂M_R, $\|u\|_{m,2} \geq c(R) > 0$, *for some constant* $c(R)$ *independent of u.*

(iii) *On* ∂M_R, $\|u\|_{0,1} \leq g(R)$, *where* $g(R)$ *is a constant independent of u.*

Theorem II (Existence Theorem). *Let G be any bounded domain* R^N. *Suppose* $f(t, x) \in Z(\rho_1)$. *Then the boundary-value problem* (1) *has a generalized eigenfunction* $u(x)$; $u(x)$ *is normalized by the requirement that* $u(x) \in \partial M_R$ *for some fixed positive R and characterized as a solution of the variational problem* inf $a(v, v)$ *for* $v \in \partial M_R$.

This result is proved by solving the above variational problem by the direct method of the calculus of variations, using the geometrical properties of ∂M_R, as illustrated in Lemma 1. We then show that a solution of the variational problem is also a solution of the operator equation.

We note the following regularity conditions associated with the boundary-value problem (1): (1a) G is of class $4m$. (1b) For the coefficients of A, $a_{\alpha\beta}(x) \in C^{2m}(\overline{G})$, (1c) $f(t, x)$ satisfies a local Lipschitz condition in t and a local Hölder condition of exponent r, $0 < r < 1$, for $x \in G$.

Theorem III (Regularity Theorem). *Suppose the regularity conditions* (1a), (1b), (1c) *for the boundary-value problem* (1) *hold, and* $f(t, x) \in Z(\rho_1)$. *Then any generalized eigenfunction for* (1) *is a classical eigenfunction.*

This result follows immediately from the results of Agmon [1], Agmon, Douglis and Nirenberg [2] and Browder [3], [4] on the L_p regularity theory of elliptic operators.

Theorem IV (Positive Eigenfunctions). *Let the hypotheses of Theorem* II *be satisfied. Then if A is a second-order operator, the boundary-value problem* (1) *has a positive generalized eigenfunction* $u(x)$ *in G. If* $u(x)$ *is a classical eigenfunction, then* $u(x) > 0$ *in G.*

Remark. The boundary-value problem studied here provides another example of nonuniqueness in the theory of quasi-linear elliptic equations of the type similar to the Navier-Stokes equation for a stationary flow of an incompressible fluid.

Added in proof (*December* 7, 1964). A variational method can also be used to prove the existence of an infinite number of distinct normalized eigenfunctions u_n with associated eigenvalues λ_n, tending to infinity with n, for the boundary value problem (I).

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