RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

SOME ARITHMETIC PROPERTIES OF THE BELL POLYNOMIALS[1]

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Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ denote indeterminates. The Bell polynomials $\phi_n(\alpha_1, \alpha_2, \alpha_3, \cdots)$ may be defined by $\phi_0 = 1$ and

$$\phi_n = \phi_n(\alpha_1, \alpha_2, \alpha_3, \cdots) = \sum \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2}\cdots} \alpha_1^{k_1}\alpha_2^{k_2}\cdots,$$

where the summation is over all nonnegative integers k_j such that

$$k_1 + 2k_2 + 3k_3 + \cdots = n.$$

For references see Bell [1] and Riordan [3, p. 36]. The general coefficient

$$A_n(k_1, k_2, k_3, \cdots) = \frac{n!}{k_1!(1!)^{k_1}k_2!(2!)^{k_2}\cdots}$$

is integral.

Some arithmetic properties of the polynomial ϕ_n have been given by Bell and some additional properties were obtained by the present writer [2]. In particular, the latter showed that

$$\phi_{pn}(\alpha_1, \alpha_2, \alpha_3, \cdots) \equiv \phi_n(\phi_p, \alpha_p, \alpha_{2p}, \cdots) \pmod{p},$$

where p is a prime and the first argument on the right is ϕ_p and not α_p.

In the present paper we consider the following problem. Let p be a fixed prime and let $\theta(n)$ denote the number of coefficients $A_n(k_1, k_2, k_3, \cdots)$ that are prime to n. Then we can state the following results.

I. Let

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$$n = p^{r_1} + p^{r_2} + \cdots + p^{r_k} \qquad (0 \leqq r_1 < r_2 < \cdots < r_k).$$

Also let B_m denote the Bell (or exponential) number defined by

$$e^{e^x - 1} = \sum_{m=0}^{\infty} B_m x^m / m!.$$

Then we have

$$(1) \qquad \theta(n) = \sum_{j=0}^{k} \sigma_j^{(k)} B_{k-j},$$

where $\sigma_j^{(k)}$ denotes the jth elementary-symmetric function of $r_1, r_2, \cdots, r_k$.

II. Let

$$n = a_1 p^{r_1} + a_2 p^{r_2} + \cdots + a_k p^{r_k},$$

where

$$0 \leqq r_1 < r_2 < \cdots < r_k, \qquad 0 < a_j < p.$$

Also let $P(n_1, n_2, \cdots, n_k)$ denote the number of unrestricted partitions of the vector $(n_1, n_2, \cdots, n_k)$ so that

$$\prod_{n_1 + \cdots + n_k > 0} (1 - x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k})^{-1} = \sum P(n_1, n_2, \cdots, n_k) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

Then we have

$$(2)\ \theta(n) = \sum_{j_s=0}^{a_s} \binom{r_1 + j_1 - 1}{j_1} \cdots \binom{r_k + j_k - 1}{j_k} P(a_1 - j_1, \cdots, a_k - j_k).$$

We remark that (2) contains (1); however the direct proof of (1) is considerably simpler than that of (2).

Proofs of these theorems and some related results will appear elsewhere.

References

1. E. T. Bell, *Exponential polynomials*, Ann. of Math. (2) **35** (1934), 258–277.

2. L. Carlitz, *Some congruences for the Bell polynomials*, Pacific J. Math. **11** (1961), 1215–1222.

3. John Riordan, *An introduction to combinatorial analysis*, Wiley, New York, 1958.

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