spread snobbism to the contrary, correct mathematics is a proper tool for obtaining physically relevant results.

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Probabilities on algebraic structures. By Ulf Grenander. Wiley, New York, 1963. 218 pp. $12.00.

The classical limit theorems of probability theory exemplify what may be termed the "large-number phenomenon." Stated roughly it is this: in combining a large number of independent random variables subject to certain "mild" restrictions, the outcome will be asymptotically either a well-determined number, or a random variable with a well-determined distribution. For example, in one version of the central limit theorem, we form $f(x_1, \cdots, x_n) = n^{-1/2}(x_1 + \cdots + x_n)$, where the x_i are independent random variables with distribution functions F_i subject only to the restrictions:

$$\int |x|^3 dF_i(x) < M < \infty, \quad \int x^2 dF_i(x) = \sigma^2, \quad \int x\, dF_i(x) = 0.$$

The conclusion is that $f(x_1, \cdots, x_n)$ is asymptotically a normal random variable with mean 0 and variance σ^2.

The variables $x_1, \cdots, x_n$ need not be combined linearly. The following instance of the large-number phenomenon, which illustrates this, has recently received attention for its possible application to nuclear physics. Suppose $x_{11}, \cdots, x_{nn}$ are n^2 random entries in a large symmetric matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Let

$$f_u(x_{11}, \cdots, x_{nn})$$

denote the proportions of the eigenvalues which do not exceed $u \max_{1 \leq i \leq n} \lambda_i$. Imposing mild restrictions on the variables $x_{11}, \cdots, x_{nn}$, we find that the functions f_u, which are highly complex functions of the variables, tend to well-determined values as $n \to \infty$. (See Chapter 7 of the book under review for more details.)

Clearly, the elucidation of the scope of this phenomenon is a major problem for probabilists. One of the motivations for studying "probabilities on algebraic structures" is that it provides a systematic approach to this problem. Namely, the instances of the large-number phenomenon that classical probability theory discovered involved addition of real-valued random variables. It is reasonable to expect that by copying its methods one can extend these results to random variables taking values in more general algebraic structures and being combined in accordance with the relevant laws of composition.

Such results may often be translated back to real-valued random variables and yield new rules of combination of random variables for which a large-number phenomenon is valid. In addition, the abstract-valued random variables may be meaningful as such, and the results so obtained may be directly applicable.

This approach can be illustrated by a simple example (which the author, perhaps, does not exploit sufficiently). Consider the real numbers as a semigroup with the binary composition $x \vee y = \max(x, y)$. One is often interested in the behavior of the maximum $x_1 \vee x_2 \vee \cdots \vee x_n$ of a set of independent random variables. To study this, let us find the analogue for this operation of the characteristic function, or Fourier transform, for ordinary addition. We notice that the "characters," or functions satisfying $\chi(x \vee y) = \chi(x)\chi(y)$, are just the functions

$$\chi_t(x) = \begin{cases} 1 & \text{for } x \leqq t \text{ (or } x < t\text{)}, \\ 0 & \text{for } x > t \text{ (or } x \geqq t\text{)}. \end{cases}$$

The characteristic function for this semigroup is then given by $\phi(t) = \int \chi_t(x)\, dF(x) = F(t)$, or the distribution function itself. Notice that $t = +\infty$ $(\chi_\infty \equiv 1)$ takes the place of the origin for ordinary characteristic functions. Having observed this, we may carry over, almost without proof, certain additive theorems to the present setup. For example, comparing with the limit laws for $a_n(x_1 + \cdots + x_n) - b_n$, where the x_i are independent and identically distributed, we can immediately infer that a nondegenerate limit distribution for a sequence $a_n \max(x_1, \cdots, x_n)$, with the x_i independent and identically distributed, must have the (stable) form $F(t) = e^{-At^{-\alpha}}$, $t > 0$.

To discuss the extent to which this program has been successful it is necessary to recall the highlights of the classical theory. For any topological group or semigroup with identity, one has the notion of an "infinitesimal" probability measure; namely, one that concentrates most of its measure close to the identity. One can then form the semigroup S of measures which are limits of convolutions $\mu_1 * \cdots * \mu_n$ of many infinitesimal measures. The basic problem of the theory is to obtain information regarding S. Now S contains the set I of infinitely divisible distributions. These are the measures which are imbeddable in a continuous one-parameter semigroup of probability measures: $\mu_{t+s} = \mu_t * \mu_s$. In the classical case one can explicitly identify the measures of I by the Lévy-Khintchine formula for the characteristic functions of measures in I. Moreover, and this is one of the triumphs of the theory, it turns out that I exhausts all of S. As a result we have a satisfactory characterization of essentially all the limit laws that can arise by additive means (more precisely, those in which the individual variables become negligible). Finally, one isolates in I certain subclasses such as the class of normal distributions, and one finds conditions for a product $\mu_1 * \cdots * \mu_n$ to converge to a distribution in one of these classes (e.g., the central limit theorems).

How much of this extends beyond the real line is still unknown. What has been shown is that the theory of infinitely divisible distributions carries over satisfactorily to Lie groups. That is, a version of the Lévy-Khintchine formula is valid, one can distinguish a class of normal distributions in I and prove a central limit theorem for any Lie group. These results, due to Hunt and Wehn, are presented in Chapter 5 of the present book. However, it is not known even for Lie groups whether I exhausts all of S (such a result would be very surprising), and nothing seems to be known as to the nature of the measures in S.

Although one cannot as yet formulate a definitive theory for general algebraic structures, a number of scattered results have been proven along these lines. These have been skillfully put together in the book under review, which represents the first large scale attempt to formulate the theory of addition of random variables in this generality. In drawing attention to the work that has been done, the author's main objective seems to have been to demonstrate the fertility of this area and to spur a systematic effort in this direction. The following quotation is a good indication of the spirit in which the book was written. "The theorems stated in the section should be taken only for what they are: preliminary and tentative endeavors to formulate a theory. Its shortcomings should be obvious to the reader and may stimulate him to try to remedy this."

The algebraic structures that are studied are compact semigroups, compact groups, abelian groups, Lie groups, locally compact groups, Banach spaces, and algebras. The author has apparently drawn some of his inspiration from the fact that harmonic analysis, which forms the basic tool of the classical theory, has, to some extent, been generalized to locally compact groups. Using irreducible unitary representations, one can not only define Fourier transforms of probability measures (or characteristic functions) for an arbitrary locally compact group, but one can prove that each measure is determined uniquely by its Fourier transform. Even more, convergence of the probability measures is equivalent to convergence of the corresponding (operator-valued) characteristic functions. As a result, limit theorems for products of group-valued random variables should, in principle, be provable just as the corresponding additive theorems for the real line. For this reason the emphasis is on harmonic analysis throughout the book with the exception of the chapter on Lie groups.

This method, however, has been notably successful only in the case of compact groups in connection with the theorem of Itô and Kawada. This states that on a compact group, an n-fold convolution μ^n of a probability measure with itself "in general" converges to the Haar measure of the group. The difficulty in the case of a general locally compact group is simply that we are not able to describe any interesting class of probability measures. (Haar measure is no longer a probability measure!) Thus, though the machinery exists for proving limit theorems in this generality, the difficulty is in formulating such theorems. In the chapter on locally compact groups the author does state results of this kind, but the apparatus of representation theory enters significantly in their formulation. It is for Lie groups that one has the means of describing various classes of probability measures and it becomes possible to state meaningful limit theorems. What has been accomplished in this case, however, has been done without benefit of harmonic analysis.

In the case of Banach spaces, the correspondence between measures and their Fourier transforms presents a number of problems which stand in the way of a straightforward generalization of the classical theory to Banach space valued variables. The characteristic function of a distribution on a Banach space is a continuous positive definite function $\phi(x^*)$ on the dual space satisfying $\phi(0)=1$. Unlike the classical case, the converse, known as Bochner's theorem, is not true and it is of interest to find an internal characterization of characteristic functions. Moreover, convergence of measures is no longer equivalent to convergence of their Fourier transforms. In the chapter on Banach spaces the author discusses some modified forms of the classical theorems and applies one of these to obtain a version of the central limit theorem in Hilbert space.

The book is highly readable and will no doubt arouse a good deal of interest in this field. The author has wisely decided not to overburden the main text with all the details necessary to make it self-contained. There is an appendix in the form of notes to the text in which some of these details are supplied and appropriate references are given. Some proofs are sketchy but, all in all, the clarity of the book is enhanced by the author's conciseness which brings the essentials of the arguments into focus.

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