SOME NEW RESULTS IN DEFINABILITY

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Consider a first-order language $\mathfrak{L}$ with identity and finitary predicate symbols. We let the letter A range over the models for $\mathfrak{L}$. Let $\mathbf{P}$, $\mathbf{Q}$ be new unary predicate symbols and let $\mathfrak{L}(\mathbf{P}, \mathbf{Q})$ be the new first-order language with the additional predicates $\mathbf{P}$ and $\mathbf{Q}$. Models for $\mathfrak{L}(\mathbf{P}, \mathbf{Q})$ will be written as (A, P, Q) where P and Q are subsets of A. Let T and S be sets of sentences of $\mathfrak{L}(\mathbf{P}, \mathbf{Q})$ (or of $\mathfrak{L}(\mathbf{P})$, or $\mathfrak{L}$). We write $T \vdash S$ to mean that every model of T is a model of S. $|X|$ shall denote the cardinal of the set X.

The following two known results are due to Beth [1] and Svenonius [4].

(I) BETH'S THEOREM. *Let T be a theory in $\mathfrak{L}(\mathbf{P})$. Then the following are equivalent.*

(i) *There exists a formula $F(t)$ of $\mathfrak{L}$ such that*

$$T \vdash \forall t(\mathbf{P}(t) \leftrightarrow F(t)).$$

(ii) *For every model A for $\mathfrak{L}$, the set*

$$X_A = \{P \mid (A, P) \text{ is a model of } T\}$$

has at most one element.

(II) SVENONIUS' THEOREM. *Let T be a theory in $\mathfrak{L}(\mathbf{P})$. Then the following are equivalent.*

(i) *There exists a finite number of formulas $F_1(t), \cdots, F_n(t)$ of $\mathfrak{L}$ such that*

$$T \vdash \bigvee_{1 \leq i \leq n} \forall t(\mathbf{P}(t) \leftrightarrow F_i(t)).$$

(ii) *For every model (A, P) of T, the set*

$$X_{A,P} = \{P' \mid (A, P') \cong (A, P)\}$$

has exactly one element P.

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Let X be a set of disjoint pairs (P, Q). Two disjoint pairs (P, Q) and (P', Q') are *separated* if $P\cap Q' = P'\cap Q = 0$. Let β be a cardinal. We say that X is *β-bounded* if X has no subset Y of power β such that any two distinct pairs in Y are not separated. In the rest of this note we let A range over *infinite* models for $\mathfrak{L}$ and α shall always denote the cardinal of the model A.

(III) MAIN THEOREM. *Let T be a theory in $\mathfrak{L}(\mathbf{P}, \mathbf{Q})$ such that $T\vdash \neg\exists t(\mathbf{P}(t)\wedge \mathbf{Q}(t))$. Then the following are equivalent.*

(i) *There exists a finite number of formulas $F_1(t, v_1, \cdots, v_m), \cdots, F_n(t, v_1, \cdots, v_m)$ of $\mathfrak{L}$ such that*

$$T \vdash \bigvee_{1\leq i\leq n} \exists v_1 \cdots v_m \forall t[(\mathbf{P}(t) \rightarrow F_i(t, v_1, \cdots, v_m)) \wedge (\mathbf{Q}(t) \rightarrow \neg F_i(t, v_1, \cdots, v_m))].$$

(ii) *For every A the set*

$$Y_A = \{(P, Q) \mid (A, P, Q) \text{ is a model of } T\}$$

is α^+-bounded.

(iii) *For every model (A, P, Q) of T, the set*

$$Y_{A,P,Q} = \{(P', Q') \mid (A, P', Q') \cong (A, P, Q)\}$$

is α^+-bounded.

(iv) *For every A, the set Y_A is 2^α-bounded.*

(v) *For every model (A, P, Q) of T, the set $Y_{A,P,Q}$ is 2^α-bounded.*

Before giving an outline of the proof of the Main Theorem, we first make two observations with a few remarks.

(a) By reading $\neg\mathbf{P}$ for $\mathbf{Q}$ in (III), we obtain the following (common) infinite analog of (I) and (II).

(IV) THEOREM. *Let T be a theory in $\mathfrak{L}(\mathbf{P})$. Then the following are equivalent.*

(i) *There exists a finite number of formulas $F_1(t, v_1, \cdots, v_m), \cdots, F_n(t, v_1, \cdots, v_m)$ of $\mathfrak{L}$ such that*

$$T \vdash \bigvee_{1\leq i\leq n} \exists v_1 \cdots v_m \forall t(\mathbf{P}(t) \leftrightarrow F_i(t, v_1, \cdots, v_m)).$$

(ii) *For every A, $|X_A| < \alpha^+$.*

(iii) *For every model (A, P) of T, $|X_{A,P}| < \alpha^+$.*

(iv) *For every A, $|X_A| < 2^\alpha$.*

(v) *For every model (A, P) of T, $|X_{A,P}| < 2^\alpha$.*

REMARKS. The equivalence of the conditions (i), (ii), (iii) in (IV) was proved by me in November, 1962. In my original proof, I required the Continuum Hypothesis (CH). In the reply (dated November 30, 1962) to my letter telling him of my result, R. Vaught showed that the CH is not needed in the theorem. Vaught's argument is rather indirect and is based on certain metamathematical considerations, assuming that a proof with the CH has already been found. He has referred to this elimination of the CH from my result in his abstract [5] and paper [6]. In the summer of 1963, I found a direct proof of my theorem without the CH. In February, 1964, I obtained the Main Theorem and hence also (IV) in its present form. I have just recently received word that M. Makkai in [3] has announced and proved the equivalence of (IV)(i) and (IV)(ii). His proof, like my earlier proof, is based on the CH. Apparently, as of the date of submission of his manuscript (April 29, 1963), he did not know that the CH is not needed or that condition (IV)(iii) can be added. At any rate, the Main Theorem (III) is clearly an improvement over all previous results along these lines.

(b) It is not necessary that the new predicates **P** and **Q** be unary. By noticing that two sets P and Q are *comparable* (i.e., either $P\subset Q$ or $Q\subset P$) if and only if

$$(P\times\overline{P})\cap(\overline{Q}\times Q)=0,$$

our Main Theorem applied to the predicates $\mathbf{P}\times\overline{\mathbf{P}}$ and $\overline{\mathbf{Q}}\times\mathbf{Q}$ yields the following.

(V) THEOREM. *Let T be a theory in $\mathfrak{L}(\mathbf{P})$. Then the following are equivalent.*

(i) *There exists a finite number of formulas $F_1(t, s, v_1, \cdots, v_m), \cdots, F_n(t, s, v_1, \cdots, v_m)$ of $\mathfrak{L}$ such that*

$$T\vdash \bigvee_{1\leqq i\leqq n} \exists v_1\cdots v_m\forall t, s[(\mathbf{P}(s)\rightarrow(F_i(t, s, v_1, \cdots, v_m)\rightarrow\mathbf{P}(t))) \wedge(\neg\mathbf{P}(s)\rightarrow(\mathbf{P}(t)\rightarrow F_i(t, s, v_1, \cdots, v_m)))].$$

(ii) *For every A, the set X_A has no subset Z of power 2^α such that any two distinct elements of Z are incomparable.*

(iii) *For every model (A, P) of T, the set $X_{A,P}$ has no subset Z of power 2^α such that any two distinct elements of Z are incomparable.*

REMARKS. It should be clear that (V) itself is not necessarily restricted to unary predicates. The finite analogs of (V), corresponding to (I) and (II), respectively, seem to be new, and we state one of them for the sake of illustration.

(VI) THEOREM. *Let T be a theory in $\mathfrak{L}(\mathbf{P})$. Then the following are equivalent.*

(i) *There exists a formula $F(t, s)$ of $\mathfrak{L}$ such that*

$$T \vdash \forall t, s[(\mathbf{P}(s) \rightarrow (F(t, s) \rightarrow \mathbf{P}(t))) \wedge (\neg \mathbf{P}(s) \rightarrow (\mathbf{P}(t) \rightarrow F(t, s)))].$$

(ii) *For every A (not necessarily infinite), the set X_A is such that any two elements of it are comparable.*

An interesting application of the other finite analog of (V), the one we have not stated, is the following. Let T be a theory in $\mathfrak{L}$ concerned with a partially ordering relation R. (This part of the hypothesis can be stated much more generally.) Suppose that every hereditary subset P of every model A of T is mapped into a subset comparable with P by every automorphism of A. Then there exists a positive integer n such that every set of pairwise incomparable elements of every model of T has at most n elements.

Before going on to the proof, we should add that all finite analogs of (III), (IV), and (V) can be proved by standard methods.

PROOF OF THE MAIN THEOREM (IN OUTLINE). To simplify matters, let us assume that $\mathfrak{L}$ is a countable language. Let $\kappa_0 = \omega$ and, for each $n < \omega$, let $\kappa_{n+1} = 2^{\kappa_n}$; let $\kappa = \sum_n \kappa_n$. It is sufficient to prove that (v) implies (i). Suppose (i) does not hold. Then by an argument using the compactness theorem, there exists a model (A, P, Q) of T such that

(1) for no formula $F(t, v_1, \cdots)$ of $\mathfrak{L}$ does the sentence

$$\exists v_1 \cdots \forall t[(\mathbf{P}(t) \rightarrow F(t, v_1, \cdots)) \wedge (\mathbf{Q}(t) \rightarrow \neg F(t, v_1, \cdots))]$$

hold in (A, P, Q).

It follows that (A, P, Q) must be infinite. We can assume, without loss of generality, that (A, P, Q) is special of power κ. (For the notion of special models, see [3].) This implies that we may assume (A, P, Q) is the union of an elementary chain

$$(A_0, P_0, Q_0) \prec \cdots \prec (A_n, P_n, Q_n) \prec \cdots$$

of elementary submodels such that for each n, (A_n, P_n, Q_n) satisfies (1) and

(2) (A_n, P_n, Q_n) is a κ_n^+-saturated model of power κ_{n+1}.

We suppose that A has been well ordered in such a way that for each n, $A_n = \{a_\beta | \beta < \kappa_{n+1}\}$. Let ${}^\kappa 2$ be the set of all functions x such that the domain of x, Dx, is some ordinal less than κ and x takes only the values 0 and 1. We assert that two functions G and H can be constructed such that the following hold.

(3) $DG = DH = {}^{\kappa}2$.

(4) If $x \subset y$, then $G(x) \subset G(y)$ and $H(x) \subset H(y)$.

(5) If $Dx < \kappa_n^+$, then $G(x)$, $H(x) \in A_n^{Dx}$ and $(A_n, G(x)) \equiv (A_n, H(x))$.

(6) Suppose $\kappa_n^+ \leqq Dx < \kappa_n^+ + \kappa_{n+1}$. Let γ be the unique ordinal less than κ_{n+1} such that $Dx = \kappa_n^+ + \gamma$. If $\gamma = \delta + 2m$ with δ a limit ordinal and $m < \omega$, then

$$G(x \cup \{\langle Dx, 0\rangle\}) = G(x \cup \{\langle Dx, 1\rangle\}) = G(x) \cup \{\langle Dx, a_{\delta+m}\rangle\}.$$

If $\gamma = \delta + 2m + 1$, with the same conditions on δ and m, then

$$H(x \cup \{\langle Dx, 0\rangle\}) = H(x \cup \{\langle Dx, 1\rangle\}) = H(x) \cup \{\langle Dx, a_{\delta+m}\rangle\}.$$

(7) Suppose $\kappa_n^+ + \kappa_{n+1} \leqq Dx < \kappa_{n+1}^+$. Let γ be the unique ordinal less than κ_{n+1}^+ such that $Dx = \kappa_n^+ + \kappa_{n+1} + \gamma$. Then there exist $p \in P_{n+1}$ and $q \in Q_{n+1}$ such that

$$G(x \cup \{\langle Dx, 0\rangle\}) = G(x) \cup \{\langle Dx, p\rangle\},$$
$$G(x \cup \{\langle Dx, 1\rangle\}) = G(x) \cup \{\langle Dx, q\rangle\},$$

and

$$H(x \cup \{\langle Dx, 0\rangle\}) = H(x \cup \{\langle Dx, 1\rangle\}).$$

The construction of G and H satisfying (3)–(7) is by transfinite induction on the ordinal Dx. It is not a difficult argument. We only mention that conditions (3)–(6) are consequences of (2), whereas condition (7) is a consequence of (1).

Suppose G and H satisfy (3)–(7). Let 2^κ be the set of all functions x such that $Dx = \kappa$ and x takes only the values 0 and 1. By (4) we can easily extend the definition of G and H to all $x \in 2^\kappa$ in such a way that $G(x)$, $H(x) \in A^\kappa$. For each $x \in 2^\kappa$, define the mapping h_x from A onto A as follows:

$$h_x(G(x)(\gamma)) = H(x)(\gamma) \qquad \text{for each } \gamma < \kappa.$$

It follows from (5) and (6) that each h_x is an automorphism of A onto A. Furthermore, suppose x, $y \in 2^\kappa$ are such that for some $n < \omega$ and γ,

$$x(\gamma) \neq y(\gamma), \qquad k_n^+ + k_{n+1} \leqq \gamma < \kappa_{n+1}^+, \quad \text{and} \quad x \restriction \gamma = y \restriction \gamma.$$

Then, by (7), the disjoint pairs (h_x^*P, h_x^*Q) and (h_y^*P, h_y^*Q) are not separated. Hence the set $Y_{A,P,Q}$ is not 2^κ-bounded and (v) fails to hold. The theorem is proved.

REMARK. It follows from the proof that every special model A has 2^α automorphisms. This result was previously known only for homogeneous universal models.

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