ORDINARY MEANS IMPLY RECURRENT MEANS

BY R. V. CHACON

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Introduction. Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space, let T be a positive linear operator from $L_1(X)$ to $L_1(X)$ whose norm is less than or equal to one. Let $\{w_k\}$, $k \geqq 1$, be a sequence of non-negative numbers whose sum is one and let $\{u_k\}$, $k \geqq 0$, be the sequence defined by $u_n = w_1 u_{n-1} + \cdots + w_n u_0$, $u_0 = 1$. Set, for any pair of functions f in $L_1(X)$ and p in $L_1(X)$, $p \geqq 0$, $Q_n(f, p) = Z_n(f)/Z_n(p)$, $Z_n(g) = \sum_0^{n-1} u_k T^k g$. Baxter, [2], [3] utilizing [6], has obtained the following result:

THEOREM 1. *The ratios $Q_n(f, p)$ have a finite limit almost everywhere on the set where $p > 0$.*

The method of proof given by Baxter is a considerable and nontrivial application of the methods given in [4]. The theorem reduces to that of [4] if one takes $w_1 = 1$, $w_k = 0$, $k \geqq 2$. The purpose of the present note is to show that the theorem of [4] yields Theorem 1 directly and in a stronger form. The stronger form of Theorem 1 gives convergence almost everywhere on the set where $\sum_0^\infty u_k T^k p > 0$ and answers a question raised in [3]. Our proof is also sufficient to yield the theorem of [1] (see [7]).

1. **Proof.** Let $(I, \mathfrak{R}, m)$ be the measure space obtained by taking I to be the positive integers, $\mathfrak{R}$ the Borel field of all subsets of I, and m the measure given by $m(\{1\}) = 1$ and, for $i \geqq 2$, by

$$m(\{i\}) = 1 - w_1 - \cdots - w_{i-1}, \quad \beta_n = w_n/(1 - w_1 - \cdots - w_{n-1}), \quad n \geqq 2, \ \beta_1 = w_1.$$

Let P be the transformation of $L_1(I)$ to $L_1(I)$ defined by left multiplication by the matrix

$$P = \begin{pmatrix} \beta_1 & 1-\beta_1 & 0 & 0 & \cdots \\ \beta_2 & 0 & 1-\beta_2 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We use P to denote the transformation and the matrix and represent the elements of $L_1(I)$ as column vectors. It follows easily that $\|P\| = 1$, that P is positive and, setting $P^n = (p_{ij}(n))$, that $u_n = p_{11}(n)$, $n \geqq 0$.[1]

Taking $(Y, \mathfrak{F}, \gamma)$ to be the direct product of $(I, \mathcal{R}, m)$ and $(X, \mathfrak{M}, \mu)$ and U the direct product of P and T, it follows that U is a positive linear operator from $L_1(Y)$ to $L_1(Y)$ and that the norm of U is less than or equal to one. We may therefore apply the ratio theorem of [4] to U with $\tilde{f}(y) = f(i, x) = \delta_{i1} \cdot f(x)$, $\tilde{p}(y) = p(i, x) = \delta_{i1} \cdot p(x)$ to obtain Theorem 1 with convergence almost everywhere on the set where $\sum_0^\infty u_k T^k p > 0$, since $U^k f(i, x) = p_{1i}(k) T^k f(x)$, $U^k p(i, x) = p_{1i}(k) T^k p(x)$ and $p_{11}(k) = u_k$.

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Brown University

[1] This is related to renewal theory. See [5] for a discussion of relevant facts.