A NOTE ON ISOMORPHISMS OF C^*-ALGEBRAS[1]

BY L. TERRELL GARDNER

Communicated by P. R. Halmos, June 16, 1964

1. **Introduction.** Let $\mathfrak{H}_i$, $i=1, 2$ be two Hilbert spaces of the same Hilbert dimension, $\mathfrak{L}(\mathfrak{H}_i)$, the algebra of all bounded linear operators on $\mathfrak{H}_i$. If S is any invertible, bounded linear mapping of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$, the mapping $A \to SAS^{-1}$ is an algebraic isomorphism (called "spatial") of $\mathfrak{L}(\mathfrak{H}_1)$ onto $\mathfrak{L}(\mathfrak{H}_2)$ which is a *-isomorphism (adjoint-preserving) if and only if S is unitary. This isomorphism ψ—or its restriction to a norm-closed *-subalgebra $\mathfrak{A}$ of $\mathfrak{L}(\mathfrak{H}_1)$ such that $\mathfrak{B}=\psi(\mathfrak{A})$ is also a norm-closed *-algebra—affords the most accessible illustration of an isomorphism of C^*-algebras which is not a *-isomorphism. Of course, the $\mathfrak{L}(\mathfrak{H}_i)$ are *-isomorphic, under some other maps—but what of $\mathfrak{A}$ and $\mathfrak{B}$? Even for W^*-algebras, the question has remained open: if $\mathfrak{A}$ and $\mathfrak{B}$ are algebraically isomorphic, are they necessarily *-isomorphic? See, e.g. [7, p. 1.53, Problem (i)].

In this note, the above question is answered affirmatively for the more inclusive class of C^*-algebras [Theorem 3].

Theorem 2 gives the structure of isomorphisms of C^*-algebras, showing that each is, in a certain canonical sense, spatial in nature. The Invariance Theorem 1 stems from the theory of analytic functions in Banach algebras, and is employed with Theorem 2 to prove Theorem 3.

The proofs will be sketched. Full details will appear elsewhere.

The author wishes to express here his gratitude to Professor Richard V. Kadison, who directed his attention to the problem of Theorem 3, and without whose advice and encouragement this work would not have been done.

2. **Preliminaries: Representation theory.** By C^*-algebra we mean an abstract complex Banach *-algebra $\mathfrak{A}$ with $\|A^*A\| = \|A^*\| \, \|A\|$ for all $A \in \mathfrak{A}$ (B^*-algebra). A representation (*-representation) of $\mathfrak{A}$ on the Hilbert space $\mathfrak{H}$ is a homomorphism (*-homomorphism) of $\mathfrak{A}$ into $\mathfrak{L}(\mathfrak{H})$, the algebra of all bounded operators on $\mathfrak{H}$. A *-representation is of norm at most 1, and its image is norm-closed. A *-representation ϕ on $\mathfrak{H}$ is *cyclic* if there exists a vector x in $\mathfrak{H}$ (cyclic vector) such that the closure $[\phi(\mathfrak{A})x]$ of $\{\phi(A)x \mid A \in \mathfrak{A}\}$ is $\mathfrak{H}$. It is *irreducible* if every

[1] This work has been partially supported by the National Science Foundation under grants NSF GP-1604 and NSF G 19022.

$x \neq 0$ in $\mathfrak{H}$ is cyclic. A *-representation is *faithful* if it is a *-isomorphism, in which case it is an isometry.

A classical theorem of Gel'fand-Neumark [2], as strengthened and elegantly set forth in [3], asserts that every C^*-algebra has a faithful *-representation as a C^*-algebra of operators on a suitable Hilbert space.

An element $A \in \mathfrak{A}$ is self-adjoint if $A = A^*$, positive if self-adjoint with positive spectrum. The positive elements form a cone in $\mathfrak{A}$, linearly spanning $\mathfrak{A}$.

A *state* of $\mathfrak{A}$ is a positive linear functional ρ with $\rho(I) = 1$. The *left kernel*, $\mathfrak{I}_\rho$, of the state ρ is the set of A in $\mathfrak{A}$ such that $\rho(A^*A) = 0$. By the Schwarz inequality for ρ, $\mathfrak{I}_\rho$ is a left ideal. $\mathfrak{A}/\mathfrak{I}_\rho$ is therefore a left $\mathfrak{A}$-module in a natural way, and the algebraic representation of $\mathfrak{A}$ on $\mathfrak{A}/\mathfrak{I}_\rho$ is denoted ϕ_ρ. We define on $\mathfrak{A}/\mathfrak{I}_\rho$ a positive-definite inner product $(A + \mathfrak{I}_\rho, B + \mathfrak{I}_\rho) = \rho(B^*A)$, and after verifying that $\phi_\rho(A)$ is bounded for each $A \in \mathfrak{A}$, we extend $\phi_\rho(A)$ to a bounded operator on $\mathfrak{H}_\rho = (\mathfrak{A}/\mathfrak{I}_\rho)^-$, the completion of the prehilbert space $\mathfrak{A}/\mathfrak{I}_\rho$ in its norm $\|\cdot\|_\rho$. We call the map thus defined on $\mathfrak{A}$ to $\mathfrak{L}(\mathfrak{H}_\rho)$ again ϕ_ρ; ϕ_ρ is a cyclic *-representation of $\mathfrak{A}$ on $(\mathfrak{A}/\mathfrak{I}_\rho)^-$, with cyclic vector $I + \mathfrak{I}_\rho$; it is called the *representation due to* ρ.

We say ρ is a *pure state* of $\mathfrak{A}$ if ρ is an extreme point of the weak-* compact, convex set of states of $\mathfrak{A}$. We denote the set of pure states of $\mathfrak{A}$ by $\mathcal{P}(\mathfrak{A})$. If and only if ρ is pure, $\mathfrak{I}_\rho$ is a maximal left ideal, ϕ_ρ is irreducible, and $\mathfrak{A}/\mathfrak{I}_\rho$ is complete in the inner-product norm defined above [5].

In [5], it is shown that the correspondence between maximal left ideals and pure states is one-one: a maximal left ideal $\mathfrak{I}$ is contained in the null space of a unique pure state, ρ, and $\mathfrak{I} = \mathfrak{I}_\rho$.

3. Positive inner automorphisms.

THEOREM 1. *Let $\mathfrak{H}$ be a Hilbert space, and let $\mathcal{S}$ be a norm-closed linear subspace of $\mathfrak{L}(\mathfrak{H})$, T a positive invertible operator in $\mathfrak{L}(\mathfrak{H})$. Then if $\mathcal{S}$ is invariant under $T^{-1} \cdot T$, $\mathcal{S}$ is invariant also under $A \to A \log T - (\log T)A$, and under $T^{-s} \cdot T^s$ for all real numbers s.*

Spectral theory defines $L = \log T$ and $T^s = \exp(sL)$, for each real s as self-adjoint operators; we put $\tau^s(A) = T^{-s}AT^s$ for A in $\mathfrak{L}(\mathfrak{H})$. In the Banach algebra $\mathfrak{L}(\mathfrak{L}(\mathfrak{H}))$, with unit element denoted by e, we compute to show that $\lim_{s \to 0} s^{-1}(e - \tau^s) = \text{ad } L$, where ad $L(A) = AL - LA$. It follows that $\tau^s = \exp(s \cdot \text{ad } L)$ [4, p. 283, Theorem 9.4.2].

Next we prove that τ $(= \tau^1)$ has a positive real spectrum, and so has a logarithm approximable by $p_n(\tau)$, (p_n) a sequence of real polynomials. Call this logarithm Λ.

The proof that ad $L=\Lambda$ employs a result of E. R. Lorch [6, p. 421], 4, Theorem 5.5.5] characterizing the periods of the exponential function in a commutative Banach algebra. From ad $L=\Lambda$ we have $p_n(\tau)\to$ad L, so that the invariance of a closed subspace $\mathcal{S}$ under τ implies the invariance of $\mathcal{S}$ under ad L, then under $\exp(s\cdot \text{ad } L)=\tau^s$ for all real s. This proves Theorem 1.

4. Isomorphism and *-isomorphism.

DEFINITION. The *atomic representation* α of a C^*-algebra is the direct sum $\oplus_{\rho\in\mathcal{P}(\mathfrak{A})}\phi_\rho$ of the representations due to pure states of $\mathfrak{A}$. α acts on $\mathcal{H}=\oplus_{\rho\in\mathcal{P}(\mathfrak{A})}\mathfrak{A}\mathcal{J}_\rho$ by $\alpha(A)((x_\rho))=(\phi_\rho(A)(x_\rho))$. α is known to be a faithful, hence isometric *-representation of $\mathfrak{A}$ [8].

THEOREM 2. *Let ψ be an algebraic isomorphism of C^*-algebra $\mathfrak{A}$ onto C^*-algebra $\mathfrak{B}$, and let α (resp. β) be the atomic representation of $\mathfrak{A}$ (resp. $\mathfrak{B}$) on the Hilbert space $\mathcal{H}$ (resp. $\mathcal{K}$). Then $\beta\psi\alpha^{-1}$ can be extended to an isomorphism of $\mathfrak{L}(\mathcal{H})$ onto $\mathfrak{L}(\mathcal{K})$ of the form $A\to SAS^{-1}$ for some S in $\mathfrak{L}(\mathcal{H}, \mathcal{K})$.*

In the proof, we make use of the fact that ψ is necessarily bounded [1, p. 15, Exercise 5]. The isomorphism ψ carries each maximal left ideal $\mathcal{J}$ of $\mathfrak{A}$ onto a maximal left ideal $\mathcal{J}'=\psi(\mathcal{J})$ of $\mathfrak{B}$, inducing a linear map $S_{\mathcal{J}}$ of the quotient space $\mathfrak{A}/\mathcal{J}$ onto $\mathfrak{B}/\mathcal{J}'$. When these quotient spaces are considered as Hilbert spaces, $S_{\mathcal{J}}$ is shown to be bounded, with $\|S_{\mathcal{J}}\|\leqq\|\psi\|$. Thus $S=\oplus\{S_{\mathcal{J}}\mid \mathcal{J}$ is a maximal left ideal of $\mathfrak{A}\}$ $=\oplus_{\rho\in\mathcal{P}(\mathfrak{A})}S_{\mathcal{J}_\rho}$ is a map in $\mathfrak{L}(\mathcal{H}, \mathcal{K})$, with $\|S\|\leqq\|\psi\|$, $\|S^{-1}\|\leqq\|\psi^{-1}\|$. We identify $\mathfrak{A}$ (resp. $\mathfrak{B}$) with its image under α (resp. β), and compute to see that $\psi(A)=SAS^{-1}$ for A in $\mathfrak{A}$. This proves the theorem.

Now let $S=VT^{(1/2)}$ be the polar decomposition of S, with V unitary in $\mathfrak{L}(\mathcal{H}, \mathcal{K})$, and $T=S^*S$ in $\mathfrak{L}(\mathcal{H})$. Then $V=ST^{-(1/2)}$. Again identify $\mathfrak{A}$ with $\alpha(\mathfrak{A})$, $\mathfrak{B}$ with $\beta(\mathfrak{B})$.

LEMMA. $V\mathfrak{A}V^*=\mathfrak{B}$.

PROOF. If $A\in\mathfrak{A}$, $VAV^*=ST^{-(1/2)}AT^{(1/2)}S^{-1}=\psi(T^{-(1/2)}AT^{(1/2)})$. It suffices to show that $T^{-(1/2)}\mathfrak{A}T^{(1/2)}=\mathfrak{A}$. By Theorem 1 this is equivalent to $T^{-1}\mathfrak{A}T=\mathfrak{A}$. But for $A\in\mathfrak{A}$, $T^{-1}AT=S^{-1}(S^*)^{-1}AS^*S=\psi^{-1}((SA^*S^{-1})^*)=\psi^{-1}(\psi(A^*)^*)\in\mathfrak{A}$.

The observation that $A\to VAV^*$ is a *-isomorphism completes the proof of

THEOREM 3. *If two C^*-algebras are algebraically isomorphic, then they are *-isomorphic.*

REMARKS. (1) Professor Kadison has pointed out to the author the following corollary to Theorem 2:

Let $\mathfrak{A}$ and $\mathfrak{B}$ act faithfully and irreducibly on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ respectively, and let the isomorphism ϕ of $\mathfrak{A}$ onto $\mathfrak{B}$ have the property that ϕ carries the annihilator $\mathfrak{I}$ in $\mathfrak{A}$ of some nonzero vector $x \in \mathfrak{H}$ onto the annihilator $\mathfrak{I}'$ in $\mathfrak{B}$ of some nonzero vector $y \in \mathfrak{K}$. Then ϕ is spatial: $\phi(A) = SAS^{-1}$ for some $S \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.

We normalize x and y, and put $S(Ax) = \phi(A)y$. S is well defined, since if $Ax = 0$, $\phi(A)y = 0$. S is linear, and is bounded: In fact, the representation of $\mathfrak{A}$ on $\mathfrak{H}$ is unitarily equivalent to that of $\mathfrak{A}$ on $\mathfrak{A}/\mathfrak{I}$, with an analogous comment for $\mathfrak{B}$ on $\mathfrak{K}$. Identifying $\mathfrak{H}$ with $\mathfrak{A}/\mathfrak{I}$, $\mathfrak{K}$ with $\mathfrak{B}/\mathfrak{I}'$ via these equivalences, we see that S as defined above is the $S_{\mathfrak{I}}$ in the proof of Theorem 2, so that $\|S\| \leqq \|\psi\|$, $\|S^{-1}\| \leqq \|\psi^{-1}\|$. Clearly, S has the desired property.

(2) Since a closed two-sided ideal in a C^*-algebra is necessarily selfadjoint [9], the structure of continuous homomorphisms of one C^*-algebra onto another is given by a trivial extension of Theorem 2.

BIBLIOGRAPHY

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, (*Algèbres de von Neumann*), Gauthier-Villars, Paris, 1957.

2. I. Gel'fand and M. Neumark, *On the imbedding of normed rings into the ring of operators in Hilbert space*, Mat. Sb. (N.S.) **12** (1943), 197–213.

3. J. G. Glimm and R. V. Kadison, *Unitary operators in C^*-algebras*, Pacific J. Math. **10** (1960), 547–556.

4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, rev. ed., Amer. Math. Soc., Providence, R. I., 1957.

5. R. V. Kadison, *Irreducible operator algebras*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 273–276.

6. E. R. Lorch, *The theory of analytic functions in normed abelian vector rings*, Trans. Amer. Math. Soc. **54** (1943), 414–425.

7. S. Sakai, *The theory of W^*-algebras* (mimeographed notes), Yale University, New Haven, Conn., 1962.

8. I. E. Segal, *Irreducible representations of operator algebras*, Bull. Amer. Math. Soc. **53** (1947), 73–88.

9. ———, *Two-sided ideals in operator algebras*, Ann. of Math. (2) **50** (1949), 856–865.

COLUMBIA UNIVERSITY