STRUCTURE THEOREM FOR COMMUTATORS OF OPERATORS

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If $\mathcal{H}$ is a separable (complex) Hilbert space, and A is a (bounded, linear) operator on $\mathcal{H}$, then A is a *commutator* if there exist operators B and C on $\mathcal{H}$ such that $A = BC - CB$. It was shown by Wintner [8] and also by Wielandt [7] that no nonzero scalar multiple of the identity operator I on $\mathcal{H}$ is a commutator, and this was improved by Halmos [5] who showed that no operator of the form $\lambda I + C$ is a commutator, where $\lambda \neq 0$ and C is a compact operator. The purpose of this note is to announce the following theorem and give some indication of its proof. Details of the results described below will appear elsewhere [2].

THEOREM. *An operator A on a separable Hilbert space $\mathcal{H}$ is a commutator if and only if A is not of the form $\lambda I + C$ where $\lambda \neq 0$ and C is a compact operator.*

This theorem furnishes the solution to several problems concerning commutators posed by Halmos in [4] and [5]. In particular it is interesting to note that the identity operator is the limit in the norm of commutators and that there exists a commutator whose spectrum consists of the number 1 alone.

INDICATION OF THE PROOF. We must show that every operator that is not of the form $\lambda I + C$, with $\lambda \neq 0$ and C compact, is a commutator. These operators fall naturally into two classes; viz., the class of compact operators, which was shown to consist entirely of commutators in [1], and the class consisting of all operators that cannot be written in the form $\lambda I + C$ for any scalar λ (0 or not) and compact C. We denote this latter class by (F), and the first problem is to obtain a more usable characterization of the operators of this class. To this end we define for an arbitrary operator T on $\mathcal{H}$ the function

$$\eta_T(x) = \|Tx - (Tx, x)x\|, \qquad x \in \mathcal{H}, \ \|x\| = 1,$$

and denote by $\eta_T(\mathcal{M})$ the supremum over the subspace $\mathcal{M} \subset \mathcal{H}$ of $\eta_T(x)$.

PROPOSITION 1. *An operator T is of type (F) if and only if* $\inf \eta_T(\mathcal{M}) > 0$ *where the infimum is taken over all cofinite-dimensional subspaces $\mathcal{M}$ of $\mathcal{H}$.*

This proposition may then be employed to yield a "standard form" for operators of type (F).

PROPOSITION 2. *Every operator of type* (F) *is similar to an operator of the form*

$$\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I \\ A_{31} & A_{32} & 0 \end{pmatrix}$$

acting in the usual fashion on a Hilbert space $\mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K}$. *(The* A_{ij} *are, of course, operators on* $\mathcal{K}$.*)*

It is easily seen from this that to complete the proof it suffices to show that every 2×2 operator matrix of the form

$$\begin{pmatrix} A & U \\ B & 0 \end{pmatrix},$$

where U is an isometry with infinite deficiency, is a commutator. This is accomplished by making a fairly intricate sequence of computations involving 2×2 matrices with operator entries. A central tool used in this argument is the result [6] that every operator with an infinite-dimensional null space is a commutator.

We note in conclusion that the restriction to separable spaces in the statement of the above theorem is for the sake of simplicity only; analogous results hold for an arbitrary infinite-dimensional Hilbert space.

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