SOME RESULTS CONCERNING COMPLETELY 0-SIMPLE SEMIGROUPS

BY GÉRARD LALLEMENT AND MARIO PETRICH

Communicated by E. Pitcher, July 21, 1964

We follow the notation and terminology of [1]. A semigroup T with zero is said to be *0-rectangular* if it has the property: if all the products at the vertices of a closed polygonal line (with a finite number of vertices) of the multiplication table are all but one equal to a nonzero element m and the remaining product is not zero, then it is also equal to m. A *rectangular 0-band* is a Rees matrix semigroup with zero over the one-element group.

THEOREM 1. *Let* $S = \mathfrak{M}^0(G; I, \Lambda; P)$. *Then the following statements are equivalent*:

(a) $S \cong G \times E / G \times \{0\}$, *where E is a rectangular 0-band*;

(b) *there exist invertible matrices U $(I \times I)$ and V $(\Lambda \times \Lambda)$ such that $Q = VPU$ is a regular matrix all of whose nonzero entries are equal to* 1;

(c) *there exist mappings* $\alpha: I \to G$, $\beta: \Lambda \to G$ *such that* $p_{\lambda i} = \beta(\lambda)\alpha(i)$ *if* $p_{\lambda i} \neq 0$;

(d) *S is 0-rectangular*;

(e) *if* $p_{\lambda_1 i_1}, p_{\lambda_1 i_2}, p_{\lambda_2 i_2}, \cdots, p_{\lambda_n i_n}, p_{\lambda_n i_1} \neq 0$, *then* $p_{\lambda_1 i_1}^{-1} p_{\lambda_1 i_2} p_{\lambda_2 i_2}^{-1} \cdots p_{\lambda_n i_n}^{-1} p_{\lambda_n i_1} = 1$;

(f) *S has a subsemigroup intersecting each $\mathcal{H}$-class of S in exactly one element.*

The semigroup in (f) need not be unique. We note that an analogous result is valid for completely simple semigroups (i.e., without zero); in such a case (b) and (c) remain essentially the same, (a) becomes $S \cong G \times E$, E is a rectangular band, in (d) "0-rectangular" is replaced by "rectangular," in (e) it suffices to take four entries of P at a time, and (f) states that idempotents form a semigroup (and thus in this case the semigroup in (f) is unique).

An ideal I of a semigroup T is said to be a *matrix ideal* of T if: for all $a, b, c \in T$, (a) $aTb \subseteq I$ implies $a \in I$ or $b \in I$, (b) $abc \in I$ implies $ab \in I$ or $bc \in I$.

THEOREM 2. *Let S be a semigroup with a completely 0-simple ideal M. In order that there exist an M-homomorphism of S onto M, it is necessary and sufficient that* (0) *be a matrix ideal of S, and the restriction to M of the finest congruence ρ on S, having* 0 *as one of its classes and such that S/ρ is a rectangular 0-band, coincides with the $\mathcal{H}$-equivalence on M.*

A corresponding result is valid for semigroups with a completely simple ideal; in such a case S/ρ in the theorem is a rectangular band (cf. [2]).

A semigroup T with zero is said to be 0-*inversive* if for every $a \in T$, $a \neq 0$, there exists $x \in T$ such that ax is a nonzero idempotent. T is *weakly* 0-*cancellable* if $xa = xb \neq 0$ and $ay = by \neq 0$ implies $a = b$ [3].

THEOREM 3. *The following conditions on a semigroup S with zero are equivalent*:

(a) *S is regular and all its nonzero idempotents are primitive*;

(b) *S is regular and for all a, $x \in S$, $axa = a \neq 0$ implies $xax = x$*;

(c) (i) *for all $a \in S$, there are $e, f \in S$ such that $a = ea = af$*,

(ii) *to each $a \in S$, $a \neq 0$, and e, f as in* (i), *corresponds a unique $a' \in S$ such that $aa' = e$, $a'a = f$*;

(d) *S is* 0-*inversive and weakly* 0-*cancellable*;

(e) *S is* 0-*inversive and every nonzero principal left* (*and right*) *ideal of S is* 0-*minimal*;

(f) *S is a mutually annihilating sum* [4] *of completely* 0-*simple semigroups.*

As a consequence of this theorem one obtains certain results concerning inverse semigroups with zero (cf. [5]). For in (a) and (b) one substitutes "regular" by "inverse" which is equivalent to uniqueness of e and f in (c), and in (f) "completely 0-simple" is replaced by "Brandt semigroup"; the remaining items also have their analogues for this case.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

2. H. Hashimoto, *On the structure of semigroups containing minimal left ideals and minimal right ideals*, Proc. Japan Acad. **31** (1955), 264–266.

3. G. Lallement, *Sur les homomorphismes d'un demi-groupe sur un demi-groupe complètement-*0*-simple*, C. R. Acad. Sci. Paris **258** (1964), 3609–3612.

4. E. S. Ljapin, *Normal complexes of associative systems*, Izv. Akad. Nauk SSSR Ser. Mat. **14** (1950), 179–192. (Russian)

5. P. S. Venkatesan, *On a class of inverse semigroups*, Amer. J. Math. **84** (1962), 578–582.

UNIVERSITY OF PARIS (C.N.R.S.) AND
LEHIGH UNIVERSITY