THE METASTABLE HOMOTOPY OF $O(n)$

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It is not easy to determine how many trivial line bundles can be split off a stable real vector bundle; the first crucial question concerns bundles over a $4k$-sphere. The following result is best possible for the stated spheres:

THEOREM 1. *A nontrivial stable real vector bundle over S^{4k} is the sum of an irreducible $(2k+1)$-plane bundle and a trivial bundle, if $k>4$.*

This theorem follows from, and implies, the following theorem. The homotopy group $\pi_q(O(n))$ is *stable* for $q<n-1$ (in which case it has been described by Bott [1]), and *metastable* for $q<2(n-1)$. Except for the special cases $n\leqq 12$ the metastable groups are related to the stable groups by

THEOREM 2. *For $q<2(n-1)$ and $n\geqq 13$,*

$$\pi_q(O(n)) = \pi_q(O) \oplus \pi_{q+1}(V_{2n,n}).$$

In fact, splitting occurs in the homotopy sequence of the fibration $O(2n)\rightarrow V_{2n,n}$ at the stated groups. The behaviour in the omitted cases is easily determined from known results.

It follows that the metastable homotopy groups of $O(n)$ exhibit a periodicity, for the second summand is a stable homotopy group of the Stiefel manifold: by [4],

$$\pi_{q+1}(V_{2n,n}) \approx \pi_{q+1}(RP^\infty/RP^{n-1}).$$

Now James has shown [2] that these have a periodicity in a natural way, and in particular that if t denotes the number of nonzero homotopy groups of O in dimensions $\leqq q-n$, then

$$\pi_{q+1}(V_{2n,n}) \approx \pi_{q+1+m-n}(V_{2m,m})$$

for all $m\geqq n$ such that $m-n$ is divisible by 2^t. This isomorphism can be induced by a map of the appropriate skeleton of $V_{2n,n}$ into $\Omega^{m-n}V_{2m,m}$, and so is similar to Bott's periodicity for the stable homotopy groups.

However, the metastable periodicity in $O(n)$ does not arise in exactly the same way as Bott's. The similarity and the difference are shown by the next theorem.

THEOREM 3. *The natural fibration* $\Omega^{8s}BSO(n)\to\Omega^{8s}BSO$ *has a cross-section over the* $(n+4s-7)$*-skeleton, but in general* $BSO(n)\to BSO$ *does not have a cross-section over skeletons of dimension* $\geqq n$.

It follows that if $q=n+4s-7$, and t (described above) is $\geqq 3$, then $\Omega^{8s}BSO(n)$ and $\Omega^{8s+2^t}BSO(n+2^t)$ have the same q-type, but $BSO(n)$ and $\Omega^{2^t}BSO(n+2^t)$ do not have the same n-type.

Complete proofs and some applications will appear later; a sketch of the proof of Theorem 1 is given below.

SKETCH PROOF. Theorem 1 is implied by

THEOREM 1*. $\pi_{4k}(BSO(n))\to\pi_{4k}(BSO)$ *is trivial if* $n\leqq 2k$, *and onto if* $k>4$ *and* $n\geqq 2k+1$.

The first part is easy. For the second part, by Bott periodicity there are homotopy equivalences

$$BSp \equiv \Omega^8 BSp \equiv \Omega^{8m+4}BSO \qquad (m \geqq 4).$$

so that there are adjoint maps

$$\beta_m\colon \Sigma^{8m+4}BSp \to BSO, \qquad \beta\colon \Sigma^8 BSp \to BSp.$$

Then β_m includes an epimorphism of homotopy groups in dimensions $\geqq 8m+8$, and factorizes into $\beta_{m-1}\circ\Sigma^{8m-4}\beta$ for $m\geqq 1$. Calculation of

$$\beta^*\colon H^{4k}(BSp; Z) \to H^{4k}(\Sigma^8 BSp; Z)$$

shows that its image is divisible by 8 if k is odd, and by 4 if k is even.

Now the fibre of $BSO(n)\to BSO(n+4)$ is $V_{n+4,4}$, and the property of β^* together with Toda's result [3] that

$$8\pi_{n+r}(V_{n+4,4}) = 0 \qquad (n \text{ odd}, r < n-1),$$

enables classical obstruction theory to prove by induction on m, with a little care,

LEMMA 4. β_m: $\Sigma^{8m+4}BSp\to BSO$ *can be deformed so as to map the* $8k$*-skeleton into* $BSO(8k+1-4m)\subset BSO$.

The analogous but more delicate result for the $(8k+8)$-skeleton is too complicated to merit description here. These results are not sharp enough to prove Theorem 1* at once; the proof is concluded by observing that the generator of $\pi_{4k}(BSp)$ can be represented by a composition

$$S^{4k} \xrightarrow{f} X \xrightarrow{g} BSp,$$

where X is a $(4k-16)$-fold suspension of the Cayley plane. The cohomology maps f^*, g^* can be computed sufficiently accurately for the proof to be completed by the same kind of obstruction argument as before.

References

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4. J. H. C. Whitehead, *Note on* $\pi_r(V_{n,m})$, Proc. London Math. Soc. (2) **48** (1944), 243–291.

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