RESEARCH PROBLEMS

10. R. M. Redheffer: *Operators on Hilbert space.*

Let u, r, s, w, z denote closed linear operators defined on a Hilbert space H, with $r \neq 0$, $s \neq 0$ and $\|w\| \leqq 1$. Define operators

$$f(z) = u + rz(1 - wz)^{-1}s, \qquad S_\lambda = \begin{pmatrix} r\lambda & u \\ w & \lambda^{-1}s \end{pmatrix}$$

on H and $H \times H$, respectively, λ being a positive scalar. As norm $\|u\|$ we take sup $|uv|$ for $v \in H$, $|v| = 1$, and similarly in other cases, such as $\|S_\lambda\|$. Lengths on $H \times H$ are related to those on H by

$$|(v_1, v_2)|^2 = |v_1|^2 + |v_2|^2, \qquad v_i \in H.$$

Problem A. Give a simple proof of the following: If $\|f(z)\| \leqq 1$ for all $\|z\| \leqq 1$ such that $(1 - wz)^{-1}$ exists, then $\|S_\lambda\| \leqq 1$ for some λ.

Problem B. Give a simple proof of this: If sup $\|f(z)\| < 1$ for $\|z\| \leqq 1$, then $f(z)$ has a fixed point in $\|z\| < 1$.

Problem C. What happens in Problem B if we only have $\|f(z)\| \leqq 1$ for $\|z\| \leqq 1$?

Problem D. Let U denote the class of unitary operators, and N the class with norm $\leqq 1$. Study the class of functions $h(z)$ that satisfy a "maximum principle" in the following sharp form:

$$\sup_{z \in N} \|h(z)\| = \sup_{z \in U} \|h(z)\|.$$

In Problems A and B the emphasis is on the word "simple." Both results have been established, but the only known proof is harder than the depth of the problems seems to warrant. I expect a simple proof because: the converse of Problem A is easy; both problems are easy when the unit ball is compact, e.g., matrices; the two problems are easily proved equivalent to each other; the appropriate form of Problem A when "$\|f(z)\| \leqq 1$ for $\|z\| \leqq 1$" is replaced by "$f(z)$ unitary for z unitary" is easy; and the fact that $f(z)$ can be written $(a + bz)(c + dz)^{-1}$ suggests connections with many well-known theories.

In Problem D the theory developed should include the known fact that $f(z)$ has the stated property when $\|w\| < 1$. (Received July 7, 1964.)

11. Solomon W. Golomb: *Random permutations.*

Let L_N be the expected length of the longest cycle in a random permutation on N letters, and let $\lambda_N = L_N/N$. (Thus, $\lambda_1 = 1$, $\lambda_2 = 3/4$, $\lambda_3 = 13/18$, $\lambda_4 = 67/96$, etc.) It is easily shown that the sequence $\{\lambda_N\}$ is monotonically decreasing, and hence a limit λ exists. Computation has shown $\lambda = .62432965 \cdots$, but nothing is known of the relationship of λ to other constants. What can be proved about the irrationality or transcendence of λ, and its relationship to classical mathematical constants? (Some nearby values *un*equal to λ include $5/8$, $1-e^{-1}$, $(5^{1/2}-1)/2$, and $\pi/5$.) (Received June 8, 1964.)

ERRATA

Robert R. Korfhage: *Correction to 'On a sequence of prime numbers.'*

It has been brought to my attention that because of the lack of an overflow check in the programming system used the factors listed for $n = 7$ are in error. Thus the value of P_8 is also wrong. Present knowledge indicates that probably $P_9 > P_8$, and thus Mullin's problem is still open. (Received July 16, 1964.)