THE RECURSIVE EQUIVALENCE TYPE OF A CLASS OF SETS[1]

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1. **Introduction.** Let us consider non-negative integers (*numbers*), collections of numbers (*sets*) and collections of sets (*classes*). The letters ϵ and o stand for the set of all numbers and the empty set of numbers respectively. We write $\subset$ for inclusion, proper or improper. A mapping from a subset of ϵ into ϵ is called a *function*; if f is a function, we denote its domain and its range by δf and ρf respectively. Let a class of mutually disjoint nonempty sets be called an *md-class*; such a class is therefore countable, i.e., finite or denumerable. We recall that the *recursive equivalence type* (abbreviated: RET) of a *set* α, denoted by $\mathrm{Req}(\alpha)$, is defined [1, p. 69] as the class of all sets which are recursively equivalent to α. We wish to consider the problem: "*How can we define the RET of an md-class in a natural manner?*" Throughout this note S stands for an md-class and σ for the union of all sets in S; for every $x \in \sigma$ we denote the unique set α such that $x \in \alpha \in S$ by α_x.

DEFINITIONS. A set γ is a *choice set* of S, if

(1) $\gamma \subset \sigma$,

(2) γ has exactly one element in common with each set in S.

The set γ is a *good* choice set of S (abbreviated: gc-set), if it also satisfies

(3) there exists a partial recursive function $p(x)$ such that $\sigma \subset \delta p$ and $(\forall x)[x \in \sigma \Rightarrow p(x) \in \gamma \cdot \alpha_x]$.

Consider the special case that the md-class S is a finite class of finite sets. Then

(a) every choice set of S is a good choice set,

(b) every two choice sets of S are recursively equivalent,

(c) every two good choice sets of S are recursively equivalent.

If the md-class S is infinite, (a) and (b) need no longer be true. For let S contain infinitely many sets of cardinality $\geqq 2$, e.g., $S = ((0, 1), (2, 3), (4, 5), \cdots)$. Then S has c choice sets. Every good choice set of S has the form $p(\sigma)$ for some partial recursive function $p(x)$, hence S has at most $\aleph_0$ good choice sets and (a) is false. Every nonzero RET contains exactly $\aleph_0$ sets; the c choice sets of S can therefore not all be recursively equivalent and (b) is false. On the

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other hand, (c) still holds. For we have

PROPOSITION P1. *Every two good choice sets of an md-class are recursively equivalent.*

Note that (a) does not even hold for every finite class consisting of two infinite sets. For let $S=(\tau, \tau')$, where τ and τ' are complementary immune sets. Then S has denumerably many choice sets, but if S had a good choice set, τ and τ' would be recursive. For every md-class S we write $\zeta(S)$ for the class of all gc-sets of S. If $\zeta(S)$ is nonempty, S is called a *gc-class*. The class (τ, τ') mentioned above is an example of an md-class which is not a gc-class. P1 enables us to give the

DEFINITION. For any gc-class S,

$$\mathrm{RET}(S) = \mathrm{Req}(\gamma), \qquad \text{for any } \gamma \in \zeta(S).$$

If S is a finite md-class of finite sets, S is a gc-class and RET(S) equals the cardinality of S. We need not exclude the trivial case that S is empty, for then $\zeta(S)$ contains exactly one set, namely o.

2. **Elementary properties.** The sets $\alpha_0, \cdots, \alpha_n$ are *separable* if there exist mutually disjoint r.e. sets $\beta_0, \cdots, \beta_n$ such that $\alpha_i \subset \beta_i$ for $0 \leqq i \leqq n$. We write $\alpha_0 | \alpha_1$ if α_0 and α_1 are separable.

PROPOSITION P2. *The finite md-class $S=(\alpha_0, \cdots, \alpha_n)$ is a gc-class if and only if $\alpha_0, \cdots, \alpha_n$ are separable; if S is a gc-class, each choice set of S is a gc-set and RET(S) equals the cardinality of S.*

A gc-class is called *isolated* if each (or equivalently, at least one) of its gc-sets is isolated. In other words, a gc-class is isolated if its RET is an isol. For every nonempty gc-class S we have: σ is a finite set if and only if S is a finite class of finite sets. Similarly,

PROPOSITION P3. *Let S be a nonempty gc-class. Then σ is an isolated set if and only if S is an isolated class of isolated sets.*

Two classes S_1 and S_2 with unions σ_1 and σ_2 respectively are *separable* if $\sigma_1 | \sigma_2$. For any two classes A and B we write

$$A \times B = \{ j(\alpha \times \beta) \mid \alpha \in A \text{ and } \beta \in B \},$$

where $j(x, y) = x + (x+y)(x+y+1)/2$.

PROPOSITION P4. *Let S_1 and S_2 be separable md-classes. Then $S_1 \cup S_2$ is an md-class and*

(a) *$S_1 \cup S_2$ is a gc-class if and only if both S_1 and S_2 are gc-classes,*

(b) *if $S_1 \cup S_2$ is a gc-class, RET$(S_1 \cup S_2) =$ RET$(S_1) +$ RET(S_2).*

PROPOSITION P5. *Let S_1 and S_2 be nonempty md-classes. Then $S_1 \times S_2$ is a nonempty md-class and*

(a) *$S_1 \times S_2$ is a gc-class if and only if both S_1 and S_2 are gc-classes,*

(b) *if $S_1 \times S_2$ is a gc-class, $RET(S_1 \times S_2) = RET(S_1) \cdot RET(S_2)$.*

3. **The class Bin(α).** Let $\{\rho_n\}$ be the canonical enumeration of the class of all finite sets [2, p. 81] and r_n = cardinality of ρ_n. For any set α and any number k we write

$$C(\alpha, k) = \{n \mid \rho_n \subset \alpha \text{ and } r_n = k\}, \quad \text{Bin}(\alpha) = \{C(\alpha, k) \mid k \geqq 1\}.$$

Note that Bin(α) is an md-class for any set α; if α is a finite set of cardinality n, the members of Bin(α) are separable and Bin(α) is a gc-class with n as cardinality and RET. For any infinite set α, Bin(α) is a denumerable md-class of infinite sets; the next proposition tells us when Bin(α) is a gc-class. We write Req(ϵ) = R and refer to [2, pp. 80, 84] for the definition of a regressive set and a regressive isol.

PROPOSITION P6. *Let α be infinite and $A = Req(\alpha)$. Then*

(a) *if α has an infinite r.e. subset, $Bin(\alpha)$ is a gc-class of RET R,*

(b) *if α is a regressive set, $Bin(\alpha)$ is a gc-class of RET A,*

(c) *if α is immune, but not regressive, $Bin(\alpha)$ is not a gc-class.*

It follows that among the c existing md-classes of immune sets, exactly c are gc-classes and exactly c are not. It is shown in [3] that though the collection Λ_R of all regressive isols is not closed under addition one multiplication, one can extend the min(x, y) function from ϵ^2 into ϵ in a natural manner to a min(X, Y) function from Λ_R^2 into Λ_R. However, min(X, Y) need no longer assume one of the values X and Y.

PROPOSITION P7. *Let α, β be two nonempty isolated sets, $A = Req(\alpha)$ $B = Req(\beta)$ and*

$$S = \{j(\xi \times \eta) \mid (\exists n)[\, n \geqq 1 \text{ and } \xi = C(\alpha, n) \text{ and } \eta = C(\beta, n)]\,\}.$$

If α and β are regressive, i.e., $A, B \in \Lambda_R$ then S is a gc-class with $RET(S)$ = min(A, B).

It can be shown that S may be a gc-class while the sets α and β are immune, but not both regressive.

4. **Characterization of gc-classes.**

DEFINITIONS. Let $p(x)$ be a partial recursive function and S a gc-class. Then $p(x)$ is a *gc-function of S*, if

(α) $\sigma \subset \delta p$ and $p(\sigma) \in \zeta(S)$,
(β) $(\forall x)[x \in \sigma \Rightarrow p(x) \in p(\sigma) \cdot \alpha_x]$,
(γ) $\rho p \subset \delta p$ and $(\forall x)[x \in \delta p \Rightarrow p^2(x) = p(x)]$.
A *gc-function* is a partial recursive function which is a gc-function of at least one gc-class.

Every gc-class has at least one gc-function. For if a partial recursive function $p(x)$ is related to S by (α) and (β), then $p(x)$ has a restriction which satisfies (α), (β) and (γ). With every partial recursive function $p(x)$ we associate the md-class $\text{Gen}(p) = \{p^{-1}(y) \mid y \in \rho p\}$ of r.e. sets. This md-class is empty if and only if $p(x)$ is nowhere defined.

PROPOSITION P8. *A partial recursive function $p(x)$ is a gc-function if and only if it satisfies (γ). Moreover, if $p(x)$ satisfies (γ), it is a gc-function of the class $S = \text{Gen}(p)$ with $\sigma = \delta p$ and $p(\sigma) = \rho p \in \zeta(S)$.*

PROPOSITION P9. *Let $p(x)$ be a gc-function of the gc-class S. Then*

$$\delta p = \sigma \Leftrightarrow S = Gen(p).$$

DEFINITION I. A class S is *primitive*, if it satisfies one of the three conditions: (i) S is empty, (ii) S is a nonempty, finite md-class of r.e. sets, (iii) S is a denumerable md-class of r.e. sets and there exists a recursive function $a(n, x)$ such that if $\alpha_n = \rho a(n, x)$, then S consists of the distinct sets $\alpha_0, \alpha_1, \cdots$.

DEFINITION II. A class S is *primitive*, if it is a gc-class with a gc-function $p(x)$ such that $S = \text{Gen}(p)$.

DEFINITION III. A class S is *primitive*, if $S = \text{Gen}(p)$ for some partial recursive function $p(x)$.

PROPOSITION P10. *The three definitions of a primitive class are equivalent.*

COROLLARY. *A class S is primitive if and only if it is a gc-class with a gc-function $p(x)$ such that $\delta p = \sigma$.*

DEFINITION. An md-class T is a *restriction* of the gc-class S, if
(a) for every $\beta \in T$, there is an α_β such that $\beta \subset \alpha_\beta \in S$,
(b) there is a $\gamma \in \zeta(S)$ such that $\beta \in T \Rightarrow \gamma \cdot \alpha_\beta \subset \beta$.

PROPOSITION P11. *An md-class is a gc-class if and only if it is a restriction of some primitive gc-class.*

While there are c gc-classes, only $\aleph_0$ of them are primitive. For each RET A there exists a gc-class with A as its RET, but a primitive class can only have one of $0, 1, \cdots, R$ as its RET. The gc-sets of a primitive class P are readily characterized. For if P is finite, the gc-sets of P are the choice sets of P, and if P is infinite, say

$$P = (\alpha_0, \alpha_1, \cdots), \qquad \alpha_n = \rho a(n, x),$$

$a(n, x)$ a recursive function, then $\gamma \in \zeta(p)$ if and only if $\gamma = \rho a(f_n, u_n)$, for a recursive permutation f_n and a recursive function u_n. Finally, the restrictions of any given primitive class can be simply described. Thus Proposition P11 serves a purpose.

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