ON THE SYMMETRY OF CONVEX BODIES

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We say that a convex body in n-dimensional Euclidean space E_n is "k-symmetric" if it coincides with its reflection through some k-plane. Let K be an n-dimensional convex body and K' a k-symmetric convex body of maximum volume contained in K. Define

$$c(K;\, k) = \frac{V(K')}{V(K)},$$

where $V(K)$ is the volume of K. Let

$$c(n,\, k) = \inf\{c(K;\, k) \colon K \subset E_n\}.$$

THEOREM 1.

$$c(n,\, k) \geqq \frac{\max\{k!,\, (n-k)!\}}{2^{n-k}n!}, \qquad 0 \leqq k < n.$$

This generalizes the result, $c(n, 0) > 2^{-n}$, proved in [3].

One can also consider K as a nonhomogeneous solid with density $f(p)$ at each $p \in K$, and ask for a symmetric subset of maximum mass. Restricting ourselves to the case of 0-symmetry (i.e., central symmetry), we define for each integrable density f on K

$$\mu(K;f) = \frac{M(K')}{M(K)},$$

where K' is a centrally symmetric convex body of maximum mass contained in K, and $M(K)$ is the mass of K. Let $\mu(K)$ be the infimum of $\mu(K;f)$, for f ranging over all integrable densities, and define

$$\mu(n) = \inf\{\mu(K) \colon K \subset E_n\}.$$

THEOREM 2. $\mu(n) \geqq 2^{-n}$, $n \geqq 3$, *and* $\mu(2) = 1/3$.

The first inequality follows from an obvious generalization of the computation of "mean symmetry" used in [3], while the second equality depends on the fact (see Theorem 4) that any plane convex body is the union of 3 centrally symmetric convex bodies.

Let $g(n)$ be the least number r such that any n-dimensional convex body K can be covered by r translates of $-K$ (equivalently, $g(n)$ is the least number r such that any n-dimensional convex body is the union of r centrally symmetric bodies). Grünbaum [2] defines the number $h(n)$ as the least number r with the following property: if $\mathfrak{F}$ is any family of pairwise intersecting translates of a convex body $K \subset E_n$, then there exist r points such that each member of $\mathfrak{F}$ contains at least one of them.

THEOREM 3. *$h(n) \geqq g(n) \geqq c(n, 0)^{-1}$, for all n.*

It is shown in [1] that

$$c(n, 0) < \sqrt{\frac{2}{\pi}} \left(\frac{2}{e}\right)^n \left(\frac{n}{n+1}\right)^{n-1} \sqrt{(n+1)}.$$

Together with Theorem 3, this implies that $h(n)$ grows faster than any fixed power of n, showing that the conjecture of [2], viz. $h(n) \leqq n+1$, is false. Indeed, the conjecture fails for $n=3$, since

$$g(3) \geqq 7.$$

The last inequality follows from the fact that a tetrahedron T in E_3 cannot be covered with fewer than 7 translates of $-T$. In E_2 we have a sharp result.

THEOREM 4. $g(2)=3$.

REFERENCES

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