CROSS-SECTIONS OF SOLUTION FUNNELS

BY CHARLES C. PUGH[1]

Communicated by L. Markus, March 30, 1964

1. Consider R^{n+1} as (t, y)-space where t is real and $y=(y^1, \cdots, y^n) \in R^n$. Let $\mathfrak{F}^n$ denote the set of all continuous maps $f: R^{n+1} \to R^n$ having compact support. For any $p=(t_0, y_0) \in R^{n+1}$ and $f \in \mathfrak{F}^n$, an f-solution through p is any C^1 map $y: R \to R^n$ such that $y(t)$ is a solution of the initial value problem

$$\frac{dy(t)}{dt} = f(t, y(t)), \qquad y(t_0) = y_0.$$

The f-funnel through p, $F(p)$, is the union of all the curves $(t, y(t))$ in R^{n+1} such that $y(t)$ is an f-solution. E. Kamke [3] introduced the term *integraltrichter* in 1932. (When f is Lipschitz continuous, then of course $F(p)$ is just the unique f-solution curve through p, but if f is only C^0 then $F(p)$ may consist of many f-solution curves.) For any real number s, the *cross-section* of $F(p)$ at time s is the set $K_s(p) = \{y \in R^n: (s, y) \in F(p)\}$.

DEFINITION. A subset A of R^m is a *funnel-section* if for some $n \geqq m$ there exist $f \in \mathfrak{F}^n$ and $p \in R^{n+1}$ such that $i(A) = K_s(p)$ for some real s, where $i: R^m \to R^n$ is the usual injection of R^m onto the span of the first m coordinate axes of R^n.

2. A theorem of H. Kneser [5] asserts that any funnel-section is a continuum (i.e., a compact, connected set). There naturally arises, then, the question: what are necessary and sufficient conditions that a continuum in R^m be a funnel-section? We prove the six theorems below as partial answers to this question.

THEOREM 1. *There exists a continuum P which is not a funnel-section.*

THEOREM 2. *There exists a funnel-section S which is not arcwise connected.*

P is a bounded outward spiral in $\mathbf{C} = R^2$ together with its limit circle:

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[1] The author holds a fellowship from the United States Steel Foundation at The Johns Hopkins University. This research was also supported in part by the Air Force Office of Scientific Research. I am glad to thank Dr. Philip Hartman for his patience and helpful advice.

$P = \{z \in \mathbf{C}: z = (1 - 1/\theta)e^{i\theta} \text{ for } 2\pi \leqq \theta < \infty\} \cup \{z \in \mathbf{C}: |z| = 1\}.$

S is a continuum of the same form as

$$\{(x, y) \in [-1, 1]^2: x \neq 0 \text{ implies } y = \sin(1/x)\}.$$

DEFINITION. A C^1 polyhedron is the image of a finite abstract polyhedron that has been imbedded in Euclidean space by a map which is bi-C^1 on each simplex.

THEOREM 3. *Any C^1 polyhedron is a funnel-section.*

Theorem 3 implies that all C^1 manifolds, algebraic varieties, and rectilinear polyhedra are funnel-sections. In particular, funnel-sections may fail to be simply connected as was previously observed by M. Nagumo and M. Fukuhara in [**8**].

3. DEFINITION. Let f be in $\mathfrak{F}^n$ and let Z be any subset of R^{n+1}. The funnel of f-solutions through Z is defined as

$$F(Z) = \bigcup_{p \in Z} F(p)$$

and the cross-section of $F(Z)$ at time s is defined to be

$$K_s(Z) = \{y \in R^n: (s, y) \in F(Z)\}.$$

This notion of the funnel of f-solutions through a set (instead of just through a point) seems first to have been defined by M. Fukuhara in [**2**]. As he points out (Theorem 2 in [**2**]), it is easy to generalize Kneser's Theorem by replacing p with a continuum Z. In our terminology, a funnel-section is the cross-section of a funnel through a point.

DEFINITION. Let f be in $\mathfrak{F}^n$ and let A be a subset of R^n. Let a and b be real numbers. We define $F(a \times A)$ to be (a, b)-stable if $A = K_a(b \times K_b(a \times A))$. (This means that if $y_0 \in A$ and if $y(t)$ is any f-solution through (a, y_0) and if $\tilde{y}(t)$ is any f-solution through $(b, y(b))$, then $\tilde{y}(a) \in A$.)

THEOREM 4. *Let f be in $\mathfrak{F}^n$ and let A be compact. Suppose that $F(a \times A)$ is (a, b)-stable for some $a, b \in R$. Then $R^n - A$ is diffeomorphic to $R^n - K_b(a \times A)$ by a diffeomorphism which is the identity on some neighborhood of infinity* (*i.e. the complement of some compact set*).

The full converse of Theorem 4 is false (which we can show by an example). However, we can prove

THEOREM 5. *Suppose A is a continuum in R^n such that $R^n - A$ is diffeomorphic to $R^n - 0$ by a diffeomorphism which is the identity on a neighborhood of infinity. Then A is a stable funnel-section, i.e., there* *exists* $f \in \mathfrak{F}^n$ *and* $y_0 \in R^n$ *such that, for* $p=(0, y_0)$, $F(p)$ *is* $(0, 1)$-*stable and* $K_1(p)=A$.

Theorems 4 and 5 completely characterize stable funnel-sections.

4. A stronger version of Theorem 5 would be

THEOREM 5'. *If* A *is a continuum in* R^n *such that* R^n-A *is diffeomorphic to* R^n-0, *then* A *is a stable funnel-section.*

To deduce Theorem 5' directly from Theorem 5, we should need the following proposition from topology.

PROPOSITION 1. *If* A *is a continuum in* R^n *such that* R^n-A *is diffeomorphic to* R^n-0, *then there is a diffeomorphism* $f: R^n-A \to R^n-0$ *which is the identity in a neighborhood of infinity.*

This proposition can be proved using the C^∞ Schönflies Conjecture which asserts that if $f: S^{n-1} \to S^n$ is a C^∞ imbedding then $(S^n, f(S^{n-1}))$ is diffeomorphic to (S^n, S^{n-1}) (where S^{n-1} is considered as the equator of S^n). The C^∞ Schönflies Conjecture is valid for $n \neq 4$, by the combined results of [**1**; **6**; **7** and **9**]; so Proposition 1 and Theorem 5' are valid for $n \neq 4$.

Theorem 2 is an immediate consequence of Theorem 5' and the Riemann Mapping Theorem because the Riemann Mapping Theorem provides a diffeomorphism between R^2-S and R^2-0 since $\hat{R}^2-S$ and $\hat{R}^2-0$ are simply connected regions on $S^2=\hat{R}^2=$ the one point compactification of R^2.

5. DEFINITION. Two subsets A and B of R^n are C^1 equivalent if there exist neighborhoods U and V of A and B and a bi-C^1 homeomorphism of U onto V which takes A onto B.

It is simple to see that if A and B are C^1 equivalent then both or neither are funnel-sections.

DEFINITION. A continuum A contained in R^n is a *small funnel-section* if there exists $f \in \mathfrak{F}^n$ and $p=(t_0, y_0) \in R^{n+1}$ such that $K_s(p)$ is C^1 equivalent to A for all $s \neq t_0$.

At first it might seem that very few continua are small funnel-sections. For instance, for $n=2$ it might seem that the circle S^1 (or any nonsimply connected continuum in R^2) could not be a small funnel-section (cf. [**8**, pp. 238–239]). However, this is not the case. We have

THEOREM 6. S^1 *is a small funnel-section.*

6. The open questions about funnel-sections include:

1. What is a necessary and sufficient condition that a continuum be a funnel-section?

2. Are all ANR's funnel-sections? (Relevant to this question are the facts that Theorem 2 shows that there exist funnel-sections which are not ANR's and that the P of Theorem 1 is not an ANR.)

3. Is the property of being a funnel-section a topological property? (It is easy to see that any continuum homeomorphic to P is not a funnel-section.)

4. If the property of being a funnel-section is topological, then is it actually just a property of homotopy type? In particular, can some continuum of the same homotopy type as P be a funnel-section?

5. Does there exist a continuum A in R^m which is an n-funnel-section for $n>m$ (i.e., $f\in\mathfrak{F}^n$ exists such that for some $p\in R^{n+1}$ and $s\in R$, $K_s(p)=i(A)$, but no such f exists in $\mathfrak{F}^m$)?

6. Does there exist a funnel-section which is not small?

7. What are some examples other than P of continua which are not funnel-sections? Is there any reason that a continuum fails to be a funnel-section other than that it "incorporates the global spiral shape of P?" Is this at least true for continua in R^2?

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JOHNS HOPKINS UNIVERSITY