# RECURRENT RANDOM WALKS WITH ARBITRARILY LARGE STEPS

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**Introduction.** The random walk generated by the distribution function (d.f.), $F$, is the sequence $S_n = X_1 + \cdots + X_n$, of sums of independent and $F$-distributed random variables. If $P\{|S_n| < 1, \text{i.o.}\} = 1$, $F$ is called recurrent.[1] If $F$ is not recurrent, $P\{|S_n| \to \infty\} = 1$ [1], and $F$ is called transient. This note contains a proof that there are recurrent distributions with arbitrarily large tails. This assertion was made without proof in [2], where it is shown that for convex distributions, such behavior cannot take place.

1. **Comparing random walks.** We shall prove the following theorem.

THEOREM. *If $\epsilon = \epsilon(x)$ is defined for $x \geqq 0$, and $\epsilon(x) \to 0$, as $x \to \infty$, then there is a* recurrent *distribution function $F$, for which, for some $x_0$,*

$$(1.1) \qquad 1 - F(x) = F(-x) \geqq \epsilon(x), \qquad x \geqq x_0.$$

This result may be restated in the following way. For any d.f. $G$, there is a recurrent d.f. $F$, and a sample space $W$ on which sequences $X_n = X_n(w)$, $Y_n = Y_n(w)$, $n = 1, 2, \cdots$, may be defined so that for each $w \in W$,

$$(1.2) \quad |Y_n(w)| < |X_n(w)|, \text{ sign } Y_n(w) = \text{sign} X_n(w), \; n = 1, 2, \cdots,$$

where $Y_n$, $n = 1, 2, \cdots$, are independently $G$-distributed, and $X_n$, $n = 1, 2, \cdots$, are independently $F$-distributed. Considering $G$ transient, we have

$$(1.3) \; P\{|Y_1 + \cdots + Y_n| \to \infty, \; |X_1 + \cdots + X_n| < 1, \text{i.o.}\} = 1$$

We remark that $F$ cannot be chosen convex. If $F$ is (eventually) convex, and $1 - F(x) = F(-x) \geqq 1 - G(x) = G(-x)$, where $G$ is transient, then $F$ is also transient [2].

The idea of the proof of the theorem is to move out the mass of $G$ and bunch it up, leaving large gaps, so that the remaining steps somehow cancel themselves out.

2. **Proof of the cancellation theorem.** For symmetric $F$, the condition that $F$ be recurrent is a tail condition [2], and may be stated

[1] i.o. or infinitely often here means for infinitely many $n = 1, 2, \cdots$.

in terms of the characteristic function, $\phi(z)=\int \cos xz\, dF(x)$, as

$$\int_0^1 (1-\phi(t))^{-1}dt = \infty. \tag{2.1}$$

Since any function $\epsilon$ of our theorem is majorized by a piecewise constant function, continuous except at integers, and decreasing to zero, we may restrict ourselves to functions of this type.

We shall prove the stronger assertion.

LEMMA. *If $p_n>0$, $n=1, 2, \cdots$, $\sum p_n<\infty$, and $0<y_n\uparrow\infty$, are given, then*

$$\int_0^1 (\sum p_n(1-\cos x_n t))^{-1}dt = \infty, \tag{2.2}$$

*for some $x_n \geqq y_n$, $n=1, 2, \cdots$.*

Assuming the lemma, choose $x_0$ so that $\epsilon(x_0^-)\leqq 1/2$, and set $p_n=\epsilon(y_n^-)-\epsilon(y_n^+)$, where $y_n$, $n=1, 2, \cdots$, are the jumps of $\epsilon$ to the right of $x_0$. We define $F$ to have mass $p_n$ at $\pm x_n$, $n=1, 2, \cdots$, provided by the lemma. The remaining mass of $F$, $1-2\epsilon(x_0^-)$ $=1-2\sum p_n$ is placed at zero. As defined, $F$ is symmetric, and

$$1-F(x) = \sum_{x_n\geqq x} p_n \geqq \sum_{y_n\geqq x} p_n \geqq \epsilon(x),$$

for $x>x_0$. By (2.2), we have (2.1), and $F$ is recurrent.

To prove the lemma, assume that $n_0=0<n_1<\cdots<n_k$ have already been defined (start at $k=0$), and that $x_1, \cdots, x_{n_k}$ have been chosen so that $x_n\geqq y_n$, $n=1, 2, \cdots, n_k$, and

$$\int_0^1 \left(\sum_{n\leqq n_k} p_n(1-\cos x_n t) + 2\sum_{n>n_k} p_n\right)^{-1} dt \geqq k. \tag{2.3}$$

We shall show that it is possible to choose $n_{k+1}>n_k$ and $x_{n_k+1}, \cdots, x_{n_{k+1}}$, so that $x_n\geqq y_n$, $n_k<n\leqq n_{k+1}$, and so that (2.3) holds with $k$ replaced by $k+1$. Having shown this, $x_n$ are then inductively defined for all $n=1, 2, \cdots$ and $x_n\geqq y_n$. Moreover, for any $k$,

$$\begin{aligned} &\int_0^1 (\textstyle\sum p_n(1-\cos x_n t))^{-1}dt \\ &\qquad \geqq \int_0^1 \left(\sum_{n\leqq n_k} p_n(1-\cos x_n t) + 2\sum_{n>n_k} p_n\right)^{-1} dt, \end{aligned} \tag{2.4}$$

and by (2.3), (2.2) follows.

We now show that $n_{k+1}=m$, and $x_{n_k+1}=x_{n_k+2}= \cdots =x_{n_{k+1}}=x$ can be defined, where $x\geqq y_{n_{k+1}}$, and $m>n_k$. This is a consequence of the following assertion, where $a=n_k$ is fixed

$$\lim_{m\to\infty}\lim_{x\to\infty}\int_0^1\Bigg(\sum_{n\leqq a} p_n(1-\cos x_n t) + \Bigg(\sum_{n=a+1}^{m} p_n\Bigg)(1-\cos xt)+2\sum_{n>m} p_n\Bigg)^{-1} dt=\infty. \tag{2.5}$$

Since $\sum_{n\leqq a} p_n(1-\cos x_n t)\leqq ct^2$ for some fixed $c>0$, we find that (2.5) is a consequence of (2.6),

$$\lim_{\epsilon\to 0}\lim_{x\to\infty}\int_0^{2\pi}(t^2+1-\cos xt+\epsilon^2)^{-1}dt=\infty. \tag{2.6}$$

Writing $\int_0^{2\pi}=\sum_{n=1}^{x}\int_{2\pi(n-1)\leqq tx<2\pi n}$, and using the fact that $1-\cos r\leqq cr^2$, for some $c>0$, we have only to show that

$$\lim_{\epsilon\to 0}\lim_{x\to\infty} x^{-1}\sum_{n=1}^{x}\int_0^{2\pi}(n^2x^{-2}+r^2+\epsilon^2)^{-1}dr=\infty. \tag{2.7}$$

Noting that $a_1^2+a_2^2+a_3^2\leqq(a_1+a_2+a_3)^2$ for $a_i\geqq 0$, $i=1, 2, 3$, and integrating, the sum in (2.7) is at least

$$x^{-1}\sum_{n=1}^{x}\int_0^1(nx^{-1}+r+\epsilon)^{-2}dr = \sum_{n=1}^{x} x^{-1}(nx^{-1}+\epsilon)^{-1}(nx^{-1}+1+\epsilon)^{-1}. \tag{2.8}$$

For $\epsilon<1$, this sum is at least $\sum_{n=1}^{x}(n+\epsilon x)^{-1}3^{-1}$. Now, as $x\to\infty$,

$$\sum_{n=1}^{x}(n+\epsilon x)^{-1}=\log x(1+\epsilon)-\log \epsilon x+O(1). \tag{2.9}$$

Hence the first limit in (2.7) is $\log 1+\epsilon^{-1}$, which, indeed, tends to infinity with $\epsilon^{-1}$. This proves the assertions.

## References

1. K. L. Chung and W. H. J. Fuchs, *On the distribution of sums of random variables*, Mem. Amer. Math. Soc. No. 6 (1951), 12 pp.

2. L. A. Shepp, *Symmetric random walk*, Trans. Amer. Math. Soc. **104** (1962), 144–153.

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