LOCALLY FLAT STRINGS

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I. **The Schoenflies Theorem for strings.** In [1], Stallings defines a *string* of type (n, k) to be a pair (R^n, Y), where Y is a closed subset of R^n such that Y is homeomorphic to R^k. Similarly, he defines a pair (S^n, X), where X is homeomorphic to S^k, to be a *knot* of type (n, k). A pair (A, X) of (n, k)-manifolds is said to be *locally smooth* if each point of X has a neighborhood U in A such that the pair $(U, U\cap X)$ is homeomorphic to the pair (R^n, R^k). Thus, his definition of locally smooth is equivalent to Brown's [2] definition of locally flat.

Let (R^n, Y) be a locally smooth string of type $(n, n-1)$; Y separates R^n into two components whose closures are A and B. In [1], Stallings states that it seems possible that either A or B must be homeomorphic to a closed half-space of R^n. Harrold and Moise [3] have proved this for $n=3$. In this note we observe that both A and B are closed half-spaces of R^n for $n>3$ and hence we have a Schoenflies theorem for strings of type $(n, n-1)$ for $n>3$.

THEOREM I.1. *Let (R^n, Y) be a locally flat string of type $(n, n-1)$ and let A and B be the closures of the complementary domains of Y in R^n. Then A and B are homeomorphic to a closed half-space of R^n for $n>3$.*

COROLLARY I.2. *Let (R^n, Y) be a locally flat string of type $(n, n-1)$ for $n>3$. Then (R^n, Y) is trivial, that is, there is a homeomorphism h of (R^n, Y) onto $(R^n, R^{n-1}\times 0)$.*

COROLLARY I.3. *Let f_1, f_2 be two locally flat embeddings of R^{n-1} as a closed subset of R^n for $n>3$. Then there is a homeomorphism h of R^n onto R^n such that $hf_1=f_2$.*

Theorem I.1 follows immediately from a recent result of Cantrell's [4]. Cantrell showed that a knot (S^n, Y) of type $(n, n-1)$ is trivial for $n>3$ provided Y is locally flat except at one point. Thus, if (R^n, X) is a locally flat string of type $(n, n-1)$ and (S^n, Y) is the one point compactification of (R^n, X), Y is locally flat except at the compactification point. Hence (S^n, Y) is trivial for $n>3$ and Theorem I.1 follows.

II. **The Slab Conjecture.** In this section we consider the relationship of locally flat strings of type $(n, n-1)$ to the Annulus Conjecture. We now state the Annulus Conjecture.

II.1. *The Annulus Conjecture.* Let S_1^{n-1}, S_2^{n-1} be two disjoint locally flat $n-1$ spheres embedded in S^n. Then the submanifold M of S^n bounded by $S_1^{n-1} \cup S_2^{n-1}$ is homeomorphic to $S^{n-1} \times [0, 1]$.

Although the Annulus Conjecture is unsolved for $n > 3$, the following theorem which is well known but does not seem to be in print holds.

THEOREM II.2. *Let S_1^{n-1}, S_2^{n-1} be two disjoint locally flat $n-1$ spheres embedded in S^n. Then if M is the submanifold of S^n bounded by $S_1^{n-1} \cup S_2^{n-1}$ and $M_i = M - S_i^{n-1}$, M_i is homeomorphic to $S^{n-1} \times [0, 1)$ for $i = 1, 2$.*

PROOF. Let A_i be the closed n-cell [2] with boundary S_i^{n-1} such that $A_i \cap M = S_i^{n-1}$ for $i = 1, 2$. A_i is cellular and hence by Theorem I of [5], S^n/A_i is homeomorphic to S^n and the theorem follows.

A theorem analogous to Theorem II.2 holds for locally flat strings of type $(n, n-1)$ for $n > 3$.

THEOREM II.3. *Let R_1^{n-1}, R_2^{n-1} be two disjoint locally flat $n-1$ planes embedded as closed subsets of R^n for $n > 3$. Then if M is the submanifold of R^n bounded by $R_1^{n-1} \cup R_2^{n-1}$ and $M_i = M - R_i^{n-1}$, M_i is homeomorphic to $R^{n-1} \times [0, 1)$ for $i = 1, 2$.*

PROOF. In view of Corollary I.2, we may assume that $R_1^{n-1} = R^{n-1} \times 0$ and $R_2^{n-1} \subset R^{n-1} \times (0, \infty)$. Let A_2 be the closed half-space (by Theorem I.1) of R^n bounded by R_2^{n-1} which does not contain R_1^{n-1}. By Theorem I.1, $R^n - A_2$ is homeomorphic to R^n and hence by the same theorem M_2 is homeomorphic to $R^{n-1} \times [0, 1)$. Similarly, M_1 is homeomorphic to $R^{n-1} \times [0, 1)$.

We now state the Slab Conjecture.

II.4. *The Slab Conjecture.* Let R_1^{n-1}, R_2^{n-1} be disjoint locally flat $n-1$ planes embedded as closed subsets of R^n. Then if M is the submanifold of R^n bounded by $R_1^{n-1} \cup R_2^{n-1}$, M is homeomorphic to $R^{n-1} \times [0, 1]$.

It should be noted that the Slab Conjecture is false in dimension 3. A counterexample can be obtained as follows. Let S_1^2 be the 2-sphere boundary of a 3-cell obtained by "swelling" a Fox-Artin arc (Example 1.2) [6]. We may assume that S_1^2 is contained in the unit 3-ball B^3 of S^3, that $S_1^2 \cap \dot{B}^3 = p$, and that S_1^2 is locally flat at each point other than p. Let $S_2^2 = \dot{B}^3$, $R_1^2 = S_1^2 - p$ and $R_2^2 = S_2^2 - p$. Then R_1^2, R_2^2 are disjoint locally flat 2-planes embedded as closed subsets of $R^3 = S^3 - p$. The 3-dimensional Slab Conjecture would imply that the closure of the complementary domain of S_1^2 in S^3 containing R_2^2 is a closed 3-cell which is a contradiction since S_1^2 is wild in S^3.

The Slab Conjecture is unsolved for $n>3$ and the following theorem indicates that it is possibly stronger than the Annulus Conjecture.

THEOREM II.5. *The Slab Conjecture implies the Annulus Conjecture for* $n>3$.

PROOF. Let S_1^{n-1}, S_2^{n-1} be disjoint locally flat $n-1$ spheres embedded in S^n. In view of Brown's theorem [2], we may assume that $S_1^{n-1}=S^{n-1}=$ the equator of S^n and S_2^{n-1} lies in the northern hemisphere of S^n. Now there is a unique $n-1$ sphere S_β^{n-1} with the following properties:

(1) S_β^{n-1} lies in the northern hemisphere of S^n.

(2) S_β^{n-1} is concentric with $S^{n-1}=$ the equator of S^n.

(3) $S_\beta^{n-1}\cap S_2^{n-1}$ is not empty.

(4) The half-open annulus bounded by $S^{n-1}\cup S_\beta^{n-1}$ but not containing S_β^{n-1} does not intersect S_2^{n-1}.

Let $p\in S_\beta^{n-1}\cap S_2^{n-1}$ and let D^{n-1} be the standard unit $n-1$ cell in S^{n-1} with center p' where p' and p lie on a great circle passing through the north pole. Let C be the cone over the base $\dot{D}^{n-1}$ with vertex p. Then $[S_1^{n-1}-\text{Int}(D^{n-1})]\cup C=S_3^{n-1}$ is a locally flat $n-1$ sphere such that $S_3^{n-1}\cap S_2^{n-1}=p$.

If we define $R_1^{n-1}=S_3^{n-1}-p$ and $R_2^{n-1}=S_2^{n-1}-p$, then R_1^{n-1}, R_2^{n-1} are disjoint locally flat $n-1$ planes embedded as closed subsets of $S^n-p=R^n$. By the Slab Conjecture, the submanifold N^n bounded by $R_1^{n-1}\cup R_2^{n-1}$ in R^n is homeomorphic to $R^{n-1}\times[0, 1]$. Hence, there is a homeomorphism h of N^n onto $R^{n-1}\times[0, 1]$ where $h(R_1^{n-1})=R^{n-1}\times 0$ and $h(R_2^{n-1})=R^{n-1}\times 1$. Since $\dot{D}^{n-1}$ is a flat $n-2$ sphere in R_1^{n-1}, $h(\dot{D}^{n-1})$ is a flat $n-2$ sphere in $R^{n-1}\times 0$. Therefore, there is a homeomorphism g of $R^{n-1}\times 0$ onto itself such that $gh(\dot{D}^{n-1})$ is the standard unit $n-2$ sphere S_1^{n-2} in $R^{n-1}\times 0$. g extends naturally to a homeomorphism G of $R^{n-1}\times[0, 1]$ onto itself by $G(x, t)=(g(x), t)$. Then $k=Gh$ is a homeomorphism of N^n onto $R^{n-1}\times[0, 1]$ such that $k(\dot{D}^{n-1})$ is the standard unit $n-2$ sphere S_1^{n-2} in $R^{n-1}\times 0$.

Consider an n-annulus $S^{n-1}\times[0, 1]=A^n$. Let $q\in S^{n-1}\times 0$ and B^{n-1} be the unit $n-1$ cell in $S^{n+1}\times 0$ with center q. Let $q'=(q, 1)\in S^{n-1}\times 1$ and C' be the cone with base $\dot{B}^{n-1}$ and vertex q'. Take F^n to be the n-cell in A^n with boundary $C'\cup B^{n-1}$ and let $L^n=A^n-[\text{Int}(F^n)\cup\text{Int}(B^{n-1})\cup q']$.

There is a homeomorphism j of $R^{n-1}\times[0, 1]$ onto L^n such that $j(R^{n-1}\times 1)=(S^{n-1}\times 1)-q'$, $j(R^{n-1}\times 0)=[(S^{n-1}\times 0)-\text{Int}(B^{n-1})]\cup(C'-q')$ and $j(S_1^{n-2})=\dot{B}^{n-1}$. Then $f=jk$ is a homeomorphism of N^n onto L^n such that $f(\dot{D}^{n-1})=\dot{B}^{n-1}$. f extends uniquely to a homeomorphism F of $N^n \cup p$ onto $L^n \cup q'$ by taking $F(p) = q'$.

Finally, let M be the submanifold of S^n bounded by $S_1^{n-1} \cup S_2^{n-1}$. Since $F(C) = C'$, F extends to a homeomorphism of M onto A^n by extending first to take D^{n-1} onto B^{n-1} and finally extending to take the n-cell bounded by $C \cup D^{n-1}$ onto F^n. Thus, M is homeomorphic to $S^{n-1} \times [0, 1]$ and the theorem is proved.

It does not seem obvious that the Annulus Conjecture implies the Slab Conjecture for $n > 3$.

References

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