ON THE REDUCTION THEORY OF VON NEUMANN[1]

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Communicated by F. Browder, January 20, 1964

1. **Introduction.** The reduction theory of von Neumann has been reformulated and modernized by many authors (cf. [1], [5]).

However, in all of them, a W^*-algebra has been considered as an operator algebra on a Hilbert space. Therefore in order to construct the reduction theory, the notion of the direct integral of Hilbert spaces has been used and it makes the theory very complicated. On the other hand, the author [6] showed that a W^*-algebra can be intrinsically characterized as a C^*-algebra with a dual structure as a Banach space; therefore it can be imagined that the reduction theory can be space-freely developed. Along this line, the author [7] gave a new approach to the reduction theory. This approach gives an exact formulation for the problem of extending the reduction theory to nonseparable cases and moreover suggests the possibility of the extension of the theory to more general Banach algebras, because of the use of general theorems in functional analysis (the Dunford-Pettis theorem [2] and Grothendieck's theorem [3]). However, the proof given in the lecture notes [7] was still complicated. In this note, we give a simple proof for the fundamental theorems in the new approach in a more general form and in addition we state some related problems.

2. First of all, we state some facts concerning the tensor product of Banach spaces. Let E and F be two Banach spaces, $E\otimes F$ the algebraic tensor product of E and F. A norm α on $E\otimes F$ is said to be a cross norm, if for every $x\in E$ and $y\in F$, $\alpha(x\otimes y)=\|x\|\,\|y\|$. $E\otimes_\alpha F$ denotes the completion of $E\otimes F$ with respect to α. The "least cross norm" λ is obtained by the natural algebraic imbedding of $E\otimes F$ into $L(E^*, F)$, where $L(E^*, F)$ is the Banach space of all bounded linear operators of E^* into F. If under this mapping, $T^u\in L(E^*, F)$ corresponds to a tensor $u=\sum_{j=1}^n x_j\otimes y_j$, then for $x^*\in E^*$

$$T^u x^* = \sum_{j=1}^n \langle x_j, x^*\rangle y_j.$$

[1] This research was partially supported by the National Science Foundation under Grant No. NSF G-25222.

We define $\lambda(u)=\|T^u\|$. λ is the least cross norm of all cross norms α having cross norms α^* as dual norms. The greatest cross norm γ is defined by $\gamma(u)=\inf \sum_{j=1}^n \|x_j\|\,\|y_j\|$, where the inf is taken over all representations of u. γ is also a cross norm and $\gamma\geqq\lambda$. (Concerning these facts we shall refer to [8].) If $L^1(\Omega, \mu)$ is the Banach space of all complex valued integrable functions on a measure space Ω with the measure μ, and E is a Banach space, then Grothendieck [3] showed that $L^1(\Omega, \mu)\otimes_\gamma E=L^1(E, \Omega, \mu)$, where $L^1(E, \Omega, \mu)$ is the Banach space of all E-valued strongly integrable functions on the measure space (Ω, μ).

If E is separable, then by the Dunford-Pettis theorem [2], for any $x\in(L^1(\Omega, \mu)\otimes_\gamma E)^*$, there is a unique E^*-valued essentially bounded weakly* measurable function $f^x(t)$ on Ω such that

$$x(\xi\otimes\eta) = \int_\Omega \langle f^x(t), \xi\rangle\eta(t)d\mu(t)$$

and

$$\text{ess. sup}\,\|f^x(t)\| = \|x\|,$$

where $\xi\in E$ and $\eta\in L^1(\Omega, \mu)$.

Under such mapping $x\rightarrow f^x$, $(L^1(\Omega, \mu)\otimes_\gamma E)^*$ is isometrically isomorphic to $L^\infty(E^*, \Omega, \mu)$, where $L^\infty(E^*, \Omega, \mu)$ is the Banach space of all E^*-valued essentially bounded weakly* measurable functions on Ω.

Therefore, by the Dunford-Pettis theorem and Grothendieck's theorem, the dual of $L^1(E, \Omega, \mu)$ is $L^\infty(E^*, \Omega, \mu)$.

Now suppose that E^* is a Banach algebra.

THEOREM 1. *Let E^* be a Banach algebra such that E is separable and the multiplication in E^* is separately $\sigma(E^*, E)$-continuous, then the Banach space $L^\infty(E^*, \Omega, \mu)$ is also a Banach algebra under the pointwise multiplication.*

PROOF. It is enough to assume that $\mu(\Omega)=1$. Take $x, y\in L^\infty(E^*, \Omega, \mu)$. For $f\in E$, put $(R_bf)(a)=f(ab)$ for $a, b\in E^*$. Since the multiplication is separately $\sigma(E^*, E)$-continuous, $R_{y(t)}f\in E$. We shall consider a vector valued function $t\rightarrow R_{y(t)}f$ on Ω. For $a\in E^*$, we have $(R_{y(t)}f)(a)=f(ay(t))=(L_af)(y(t))$, where $(L_af)(b)=f(ab)$ for $b, a\in E^*$. Hence $(R_{y(t)}f)(a)=\langle y(t),\ g(t)\rangle$, where $g(t)=L_af$ for all $t\in\Omega$. Since $g(t)\in L^1(E, \Omega, \mu)$, $(R_{y(t)}f)(a)$ is measurable for all $a\in E^*$, so that $t\rightarrow R_{y(t)}f$ is weakly measurable. Since E is separable, $t\rightarrow R_{y(t)}f$ is strongly measurable. Hence for $\eta\in L^1(\Omega, \mu)$, $t\rightarrow\eta(t)R_{y(t)}f$ is strongly measurable. Since the set of all finite linear combinations of E-valued functions of the forms $\eta(t)f$ is norm-dense in $L^1(E, \Omega, \mu)$ and the function $\eta(t)R_{y(t)}f$

belongs to $L^1(E, \Omega, \mu)$, there is a sequence (h_n) for $h \in L^1(E, \Omega, \mu)$ such that $h_n \rightarrow h$ in $L^1(E, \Omega, \mu)$ and $R_{y(t)}h_n(t)$ is an E-valued function belonging to $L^1(E, \Omega, \mu)$ for all n. Then $\|R_{y(t)}h_n(t) - R_{y(t)}h(t)\| \leqq \|y\| \, \|h_n(t) - h(t)\|$. Since there is a subsequence (n_j) of (n) such that $\|h_{n_j}(t) - h(t)\| \rightarrow 0$ a.e., the above inequality implies the strong measurability of the function $R_{y(t)}h(t)$; since $\|R_{y(t)}h(t)\| \leqq \|y\| \, \|h(t)\|$, it belongs to $L^1(E, \Omega, \mu)$, so that $\langle x(t), R_{y(t)}h(t) \rangle = \langle x(t)y(t), h(t) \rangle$ is measurable.

Therefore $x(t)y(t)$ is weakly* measurable and $\|x(t)y(t)\| \leqq \|x(t)\| \, \|y(t)\|$. This completes the proof.

Now we shall show some examples.

Let M be a W^*-algebra whose associated space M_* is separable, then $(M_*)^* = M$ and the multiplication in M is separately $\sigma(M, M_*)$-continuous, so that $L^\infty(M, \Omega, \mu)$ is a Banach algebra; moreover we can easily show that it is a C^*-algebra (in fact, $x \in L^\infty(M, \Omega, \mu)$ implies $t \rightarrow x(t)^*$ is weakly* measurable, because $\langle x(t)^*, \eta(t)f \rangle = [\langle x(t), [\eta(t)]^- f^* \rangle]^-$, where $\eta(t) \in L^1(\Omega, \mu)$, $f \in M_*$; hence we define $x^*(t) = x(t)^*$, then $x^* \in L^\infty(M, \Omega, \mu)$ and moreover $\|x^*x\| = \text{ess. sup } \|x(t)^*x(t)\| = \text{ess. sup } \|x(t)\|^2 = \|x\|^2$).

Hence we have

Corollary 1. *Let M be a W^*-algebra with the separable associated space M_*, then $L^\infty(M, \Omega, \mu)$ is a W^*-algebra under the pointwise multiplication and its associated space is $L^1(M_*, \Omega, \mu)$.*

Next, let R be a separable reflexive Banach space, N be a weakly closed Banach algebra of bounded operators on R and $B(R)$ be the Banach algebra of all bounded operators on R, then $B(R)$ is the dual of $R \otimes_\gamma R^*$ and N is a $\sigma(B(R), R \otimes_\gamma R^*)$-closed subalgebra of $B(R)$; therefore N is a Banach algebra such that there is a separable Banach space E as follows: $E^* = N$ and the multiplication in N is separately $\sigma(N, E)$-continuous, so that $L^\infty(N, \Omega, \mu)$ is a Banach algebra such that $L^\infty(N, \Omega, \mu) = (L^1(E, \Omega, \mu))^*$.

Problem 1. Can we drop the assumption that the multiplication is separately $\sigma(E^*, E)$-continuous in Theorem 1?

Problem 2. Can we drop the assumption of separability in Theorem 1?

Problem 2 is extremely interesting in the case of "$E^* = B(\mathfrak{H})$," where $B(\mathfrak{H})$ is the W^*-algebra of all bounded operators on a nonseparable hilbert space $\mathfrak{H}$. It is closely related to the problem as to whether we can extend the reduction theory to the nonseparable case [7].

Now we shall apply the above result for the reduction theory.

Let B be the W^*-factor of type I_n $(n\leqq\aleph_0)$, then its associated space B_* is separable, so that $L^\infty(B, \Omega, \mu)$ is a W^*-algebra and $L^1(B_*, \Omega, \mu)$ is the associated space of $L^\infty(B, \Omega, \mu)$.

Now we shall show

THEOREM 2. *$L^\infty(B, \Omega, \mu)$ is a homogeneous type I_n W^*-algebra with the center $L^\infty(\Omega, \mu)\cdot 1$, where 1 is the identity of B.*

PROOF. We shall prove the case of $n=\aleph_0$, because other cases are analogous.

For $a\in B$, we define $\tilde{a}(t)=a$ for all $t\in\Omega$, then $\tilde{a}\in L^\infty(B, \Omega, \mu)$. If $(e_i|i=1, 2, \cdots)$ be a maximal family of orthogonal minimal projections in B, then $\tilde{e}_i(t)x(t)\tilde{e}_i(t)=\lambda_i(t)e_i$ for $x\in L^\infty(B, \Omega, \mu)$ and $\lambda_i(t)\in L^\infty(\Omega, \mu)$. Hence $\tilde{e}_i$ is an abelian projection and clearly $\tilde{e}_i\tilde{e}_j=0$ for $i\neq j$.

Moreover for $\eta\in L^1(\Omega, \mu)$ and $f\in M_*$ $(f\geqq 0)$, the vector valued function $\phi(t)=\eta(t)f$ belongs to $L^1(B_*, \Omega, \mu)$. Then

$$\left\langle \sum_{i=1}^{\infty}\tilde{e}_i, \phi\right\rangle = \sum_{i=1}^{\infty}\langle\tilde{e}_i, \phi\rangle = \sum_{i=1}^{\eta}\int_{\Omega}\langle e_i, \phi(t)\rangle d\mu(t)$$
$$= \sum_{i=1}^{\infty}\int\langle e_i, f\rangle\eta(t)d\mu(t).$$

Since $f(e_i)\geqq 0$ and $\sum_{i=1}^{\infty}f(e_i)=f(\sum_{i=1}^{\infty}e_i)=f(1)$, by the dominated convergence theorem,

$$\left\langle \sum_{i=1}^{\infty}\tilde{e}_i, \phi\right\rangle = \int\langle 1, f\rangle\eta(t)d\mu(t) = \langle\tilde{1}, \phi\rangle,$$

where 1 is the identity of B.

Since every element of M_* is a linear combination of positive normal elements, all finite linear combinations of the forms ϕ are norm-dense in $L^1(B_*, \Omega, \mu)$. Hence we have $\sum_{i=1}^{\infty}\tilde{e}_i=\tilde{1}$. Now let v_i be a partial isometry of B such that $v_i^*v_i=e_1$ and $v_iv_i^*=e_i$ for $i=1, 2, \cdots$. Then $\tilde{v}_i^*\tilde{v}_i=\tilde{e}_1$ and $\tilde{v}_i\tilde{v}_i^*=\tilde{e}_i$. Therefore $(\tilde{e}_i|i=1, 2, \cdots)$ is a maximal family of orthogonal equivalent maximal abelian projections in $L^\infty(B, \Omega, \mu)$.

Let Z be the center of $L^\infty(B, \Omega, \mu)$ and $z\in Z$. Then $\tilde{a}(t)z(t)=az(t)=z(t)a$ a.e. Let (a_n) be a family of elements in B which is $\sigma(B, B_*)$-dense in B, then we have $a_nz(t)=z(t)a_n$ for $t\in\Omega-\Gamma$ and all $n(\mu(\Gamma)=0)$. Hence $z(t)\in$ the center of B for $t\in\Omega-\Gamma$, so that $Z\subseteq L^\infty(\Omega, \mu)\cdot 1$. The converse is clear. Hence we have proved that $L^\infty(B, \Omega, \mu)$ is a homogeneous type $\mathrm{I}_{\aleph_0}$ W^*-algebra with the center $L^\infty(\Omega, \mu)\cdot 1$. This completes the proof.

Now let M be a W^*-algebra containing the identity on a separable Hilbert space $\mathfrak{H}$, M' the commutant of M, $R(M, M')$ the W^*-algebra generated by M and M', then $R(M, M')$ is of type I.

By the structure theorem of type I W^*-algebras, $R(M, M')$ can be uniquely decomposed into a direct sum of homogeneous type I W^*-algebras [4]. Therefore, it is enough to assume that $R(M, M')$ is a homogeneous type I_n W^*-algebra ($n \leqq \aleph_0$). If Z is the center of M, then we can express it as $Z = L^\infty(\Omega, \mu)$, because Z is a commutative W^*-algebra, so that by Theorem 2, $L^\infty(B, \Omega, \mu)$ is a homogeneous type I_n W^*-algebra with the center Z, where B is the type I_n W^*-factor.

Two homogeneous type I_n W^*-algebras are mutually $*$-isomorphic, if and only if their centers are mutually isomorphic [4], so that we have the representation $R(M, M') = L^\infty(B, \Omega, \mu)$, because the center of $R(M, M') = Z$.

If $x \in R(M, M')$, then x is considered as a B-valued essentially bounded weakly* measurable function on Ω which we express by $x = \int_\Omega x(t) d\mu(t)$. Then we can easily reproduce all the theorems in the reduction theory of von Neumann [7].

Moreover we have the exact formulation for the problem of extending the reduction theory to nonseparable cases.

Namely, the following problems are important for the reduction theory of the nonseparable case:

PROBLEM 3. Can we conclude that $L^\infty(B, \Omega, \mu)$ is a W^*-algebra for a type I_n factor B ($n > \aleph_0$)?

Let N be a homogeneous type I_n W^*-algebra ($n > \aleph_0$) with the center Z. Put $Z = L^\infty(\Omega, \mu)$. Then we still can show that $N_* = L^1(B_*, \Omega, \mu)$, where N_* is the associated space of N and B_* is the associated space of a type I_n W^*-factor B [7].

Therefore, we have

PROBLEM 4. Can we conclude $L^\infty(B, \Omega, \mu) = (L^1(B_*, \Omega, \mu))^*$, at least, as Banach spaces?

Since $N_* = L^1(B_*, \Omega, \mu)$, the dual of $L^1(B_*, \Omega, \mu)$ is isometrically isomorphic to the W^*-algebra N.

Therefore we have

PROBLEM 5. Can we characterise the dual of $L^1(B_*, \Omega, \mu)$ as some family of B-valued functions or B-valued measures on Ω?

REMARK. Our theory has powerful applications in group representation theory—in fact, the decomposition theory of representations can be completely reduced to the decomposition theory of positive functionals. Then for a positive $\phi \in L^1(B_*, \Omega, \mu)$, we can easily show that $\phi(t) \geqq 0$ for μ-a.e. $t \in \Omega$. Moreover, by the standard process of measure theory, for a uniformly separable C^*-algebra A which is $\sigma(M, M_*)$-dense in M, we can easily choose $\phi(t)$ as follows: $\phi(t)$ defines a positive linear functional of A for μ-a.e. $t \in \Omega$; the *-representation of A constructed via $\phi(t)$ is a factor-representation for μ-a.e. $t \in \Omega$.

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