SOME CURIOUS INVOLUTIONS OF SPHERES

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Consider an involution T of the sphere S^n without fixed points. Is the quotient manifold S^n/T necessarily isomorphic to projective n-space? This question makes sense in three different categories. One can work either with topological manifolds and maps, with piecewise linear manifolds and maps, or with differentiable manifolds and maps.

For $n \leqq 3$ the statement is known to be true (Livesay [6]). In these cases it does not matter which category one works with. On the other hand, for $n=7$, in the differentiable case, the statement is known to be false (Milnor [10]).

This note will show that, in the piecewise linear case, the statement is false for all $n \geqq 5$. Furthermore, for $n=5$, 6, we will construct a differentiable involution $T\colon S^n \to S^n$ so that the quotient manifold is not even piecewise linearly homeomorphic to projective space. Our proofs depend on a recent theorem of J. Cerf.

Let us start with the exotic 7-sphere M_3^7 as described by Milnor [7]. This differentiable manifold M_3^7 is defined as the total space of a certain 3-sphere bundle over the 4-sphere. It is known to be homeomorphic, but not diffeomorphic, to the standard 7-sphere.

Taking the antipodal map on each fibre we obtain a differentiable involution $T\colon M_3^7 \to M_3^7$ without fixed points. (The quotient manifold M_3^7/T can be considered as the total space of a corresponding projective 3-space bundle over S^4.) The following lemma was pointed out to us, in part, by P. Conner and D. Montgomery.

LEMMA 1. *There exists a differentiably imbedded 6-sphere, $S_0^6 \subset M_3^7$, which is invariant under the action of T, and a differentiably imbedded $S_0^5 \subset S_0^6$ which is also invariant.*

Thus in this way one constructs a differentiable involution of the standard sphere in dimensions 5, 6.

The proof will depend on the explicit description of M_3^7 (or more generally of M_k^7) which was given in [7]. Take two copies of $R^4 \times S^3$ and identify the subsets $(R^4-(0)) \times S^3$ under the diffeomorphism

$$(u, v) \to (u', v') = (u/\|u\|^2, u^h v u^j/\|u\|),$$

using quaternion multiplication, where $h+j=1$, $h-j=k$. The involution T changes the sign of v and v'. Let S_0^6 be the set of all points of M_k^7 such that $\Re(v') = \Re(uv) = 0$, where $\Re(uv) = \Re(vu)$ denotes the real part of the quaternion uv. This set is clearly invariant under T. To prove that S_0^6 is a manifold diffeomorphic to the standard 6-sphere, consider the function $g: M_k^7 \to R$ which is defined by

$$g = \Re(uv)/(1 + \|u\|^2)^{1/2} = \Re(v')/(1 + \|u'\|^2)^{1/2}.$$

It is easily verified that g is well defined, differentiable, and has only two critical points, both nondegenerate. Hence the set of zeros of g is diffeomorphic to S^6. (Compare [7].) But this set of zeros is precisely S_0^6.

Similarly let S_0^5 be the set of points of S_0^6 which satisfy $\Re(v) = \Re(u'(v')^{-1}) = 0$. This is a sphere, since it is the set of zeros of the function $f: S_0^6 \to R$ which is defined (as in [7]) by

$$f = \Re(v)/(1 + \|u\|^2)^{1/2} = \Re(u'(v')^{-1})/(1 + \|u'\|^2)^{1/2}.$$

This function also is nondegenerate, with two critical points, which completes the proof.

REMARK. It would be interesting to know whether this game could be continued one stage further, however the authors do not know any further suitable functions.

LEMMA 2. *The manifold S_0^n/T is not diffeomorphic to the projective space P^n for $n = 5, 6$.*

PROOF. Note that a tubular neighborhood of S_0^6/T in M_3^7/T can be considered as a twisted line-segment bundle over S_0^6/T. The complement of such a neighborhood is a 7-disk. (This is easily proved using the function g.) Hence the differentiable manifold M_3^7/T can be reconstructed out of S_0^6/T as follows:

STEP 1. Take the unique twisted line-segment bundle B over S_0^6/T.

STEP 2. Form a closed 7-manifold by matching the boundary of B with the boundary of a 7-disk under a certain diffeomorphism h.

Similarly, if one starts with P^6 and applies this construction, using an analogous diffeomorphism h' then one arrives at a manifold diffeomorphic to P^7. The only ambiguity here lies in the choice of h'. If one uses the wrong diffeomorphism then one will arrive instead at a manifold which is diffeomorphic to the connected sum $P^7 \# \Sigma$ for some twisted 7-sphere Σ. (Compare for example [9].)

Now suppose that S_0^6/T is diffeomorphic to P^6. Proceeding as above, it follows that M_3^7/T is diffeomorphic to some $P^7 \# \Sigma$. Passing to the 2-fold covering space, it follows that M_3^7 is diffeomorphic to $S^7 \# \Sigma \# \Sigma$.

But the group Γ_7, consisting of all oriented diffeomorphism classes of twisted 7-spheres, is cyclic of order 28 (see [5]) and the class of M_3^7 can be taken as a generator of this group (see [2]). Thus the class of M_3^7 cannot be divisible by two. This yields a contradiction, and completes the proof for $n=6$.

Now suppose that S_0^5/T were diffeomorphic to P^5. Then a similar argument would show that S_0^6/T must be diffeomorphic to $P^6 \# \Sigma'$ for some twisted 6-sphere Σ'. But every twisted 6-sphere is diffeomorphic to S^6 (see [5]). Therefore we can cancel Σ' and obtain a contradiction, which completes the proof of Lemma 2.

Choose C^1-triangulations of S_0^n/T and of P^n for $n=5, 6$. (See for example [12].) The resulting simplicial complexes will be denoted[1] by $\mathbf{S}_0^n/\mathbf{T}$ and $\mathbf{P}^n$ respectively. The two-fold covering complex $\mathbf{S}_0^n$ of $\mathbf{S}_0^n/\mathbf{T}$ is clearly a combinatorial n-sphere, and $\mathbf{T}: \mathbf{S}_0^n \to \mathbf{S}_0^n$ is a fixed point free simplicial involution.

THEOREM 1. *The complex $\mathbf{S}_0^n/\mathbf{T}$ is not piecewise linearly homeomorphic to $\mathbf{P}^n$, for $n=5, 6$.*

PROOF. Suppose that $\mathbf{P}^n$ were piecewise linearly homeomorphic to $\mathbf{S}_0^n/\mathbf{T}$. Then according to Munkres [11] there would exist a sequence of obstructions

$$\mathfrak{o}_i \in \mathcal{H}_i(\mathbf{P}^n; \Gamma_{n-i})$$

to finding a diffeomorphism between S_0^n/T and P^n. Here $\mathcal{H}_i$ denotes homology based on infinite chains, with twisted coefficients in the nonorientable case. The group Γ_m, consisting of all oriented diffeomorphism classes of twisted m-spheres, is known to be zero for $m=1, 2, 3, 5, 6$. (See [11], [5].) Furthermore, an eagerly awaited paper by J. Cerf will prove that $\Gamma_4=0$. Assuming this theorem of Cerf, it follows that all of the groups $\mathcal{H}_i(\mathbf{P}^n; \Gamma_{n-i})$ are zero for $n \leqq 6$. Thus there are no obstructions: the existence of a piecewise linear homeomorphism would imply the existence of a diffeomorphism, and hence would contradict Lemma 2. This completes the proof.

In dimension 7 our result will be somewhat weaker, since M_3^7 is not a standard 7-sphere. Choose a C^1-triangulation of M_3^7/T, thus yielding a simplicial complex $\mathbf{M}_3^7/\mathbf{T}$.

THEOREM 2. *The complex $\mathbf{M}_3^7/\mathbf{T}$ is not piecewise linearly homeomorphic to $\mathbf{P}^7$. However its 2-fold covering complex $\mathbf{M}_3^7$ is piecewise linearly homeomorphic to $\mathbf{S}^7$.*

The proof is similar to that of Theorem 1, but there is a complication since $\Gamma_7 \neq 0$. To get around this, we first remove a point x from

[1] In general,boldface letters will be used for simplicial complexes, and for piecewise linear maps.

M_3^7/T, and note that $M_3^7/T-x$ is not diffeomorphic to P^7-y. For if it were diffeomorphic, then the boundary of a small ball around x would correspond to a sphere around y bounding a manifold which, according to Smale [13, §5.1] must be a 7-disk. It would then follow that M_3^7 must be diffeomorphic to the connected sum $P^7 \# \Sigma$ for some twisted sphere Σ. But this is impossible, as we have seen during the proof of Lemma 2.

Now suppose that the corresponding complex $M_3^7/T-x$ were piecewise-linearly homeomorphic to P^7-y. Since the groups $\mathcal{H}_i(P^7-y;\ \Gamma_{7-i})$ are all zero, this would imply the existence of a diffeomorphism. We have just seen that this is impossible.

It follows a fortiori that M_3^7/T cannot be piecewise linearly homeomorphic to P^7.

Proof that M_3^7 is piecewise linearly homeomorphic to S^7 (following [8]). Recall that M_3^7 can be expressed as the union of two smooth 7-disks which intersect only along their common boundary. Choosing a suitable C^1-triangulation it follows that the resulting simplicial complex M_3^7 can be expressed as the union of two combinatorial 7-cells which intersect only along their common boundary. It now follows easily that M_3^7 is piecewise linearly homeomorphic to the combinatorial sphere S^7, which completes the proof of Theorem 2.

In still higher dimensions, one can generate examples as follows. Suppose that we start with any piecewise linear manifold Q^n whose 2-fold covering space $\hat{Q}^n$ is piecewise linearly homeomorphic to S^n. Let $Q^{n+1}=Q^n \cup C\hat{Q}^n$ denote the complex formed from Q^n by adjoining the cone over its 2-fold covering space. Then Q^{n+1} is again a piecewise linear manifold, and its 2-fold covering is the suspension of $\hat{Q}^n$. This construction can be iterated ad infinitum.

Now start with $Q^5=S_0^5/T$. It is easily verified that the corresponding Q^6 and Q^7 can be identified with S_0^6/T and M_3^7/T respectively. Each of these piecewise linear manifolds can be given a compatible differentiable structure. But if we iterate the construction once more, we obtain a piecewise linear manifold Q^8 which cannot be given a compatible differentiable structure. This can be proved using the obstruction theory of Hirsch [4]. In fact the obstruction class in

$$H^8(Q^8;\ \Gamma_7) \cong \Gamma_7/2\Gamma_7$$

can be identified with the class of the manifold M_3^7. Similarly *none of the Q^n, $n \geqq 8$, possess compatible differentiable structures.* It follows that no Q^n is piecewise linearly homeomorphic to P^n.

For each $n \geqq 5$ we have the following:

Unsolved Problem. Is the manifold Q^n homeomorphic to P^n?

The existence of such a homeomorphism would contradict the Hauptvermutung for manifolds. Its nonexistence would imply that the corresponding involution of $\hat{Q}^n \cong S^n$ is not conjugate to the antipodal map even in the group of all homeomorphisms of the n-sphere.

In conclusion let us study the extent to which our various imitation projective spaces resemble the true projective space. The following is well known.

LEMMA 3. *For any continuous fixed point free involution of a topological n-sphere, the orbit space S^n/T has the homotopy type of P^n.*

PROOF. We will construct a map $f\colon S^n \to S^n$ of degree one such that $fT(x) = -f(x)$. It is easy to see that such an f gives rise to a map $S^n/T \to P^n$ which induces isomorphisms of homotopy groups, and hence is a homotopy equivalence.

We think of S^n as the unit sphere in R^{n+1}. Define

$$f(x) = (x - Tx)/\|x - Tx\|.$$

As a parameter s runs from 0 to 1, let $T_s(x)$ run from $T(x)$ to $-x$ along the unique shortest circular arc on S^n. This arc avoids x because $T(x) \neq x$. Now define

$$f_s(x) = (x - T_s(x))/\|x - T_s(x)\|.$$

We have defined a homotopy between f and the identity proving that f has degree 1.

REMARK. In the piecewise linear case we can even assert that the orbit space S^n/T has the same *simple homotopy type* as P^n. This is true since simple homotopy type and homotopy type coincide for complexes with fundamental group of order ≤ 4. (Whitehead [14], Higman [3].)

Now let us look at the differentiable cases. Here one has an additional invariant: the tangent bundle.

LEMMA 4. *The homotopy equivalences $P^6 \to S_0^6/T$ and $P^7 \to M_3^7/T$ can be extended to bundle maps of the respective tangent bundles.*

PROOF. Let τ denote the tangent bundle of P^n, and let τ' denote the bundle over P^n induced from the other tangent bundle by the homotopy equivalence. We must prove that τ is isomorphic to τ' for $n = 6, 7$.

It follows from Adams [1, §7.4] that a vector bundle over a projective space of dimension ≤ 8 is determined up to s-isomorphism by its Stiefel-Whitney classes. But, for tangent bundles, these are homotopy type invariants. Thus τ is s-isomorphic to τ'.

Restricting to P^{n-1} it follows easily that $\tau | P^{n-1}$ is isomorphic to $\tau' | P^{n-1}$. Choosing a fixed isomorphism i between these bundles the obstruction to extending i over P^n is now a well-defined element of $H^n(P^n; \pi_{n-1}SO_n)$. For $n=7$ this group is zero, so that there is no problem.

For $n=6$ the group $H^6(P^6; \pi_5 SO_6)$ is infinite cyclic. (The coefficients are twisted.) Furthermore the projection $p: S^6 \to P^6$ induces a monomorphism $H^6(P^6; \pi_5 SO_6) \to H^6(S^6; \pi_5 SO_6)$. Hence it is sufficient to check that the obstruction becomes zero when we pass to the universal covering space S^6. But $p^*\tau$ is clearly isomorphic to $p^*\tau'$. (Both bundles have Euler number ± 2.) Hence, if i is chosen carefully, the obstruction to extending i will be zero. This completes the proof.

Added in proof. The corresponding statement for $n=5$ is true also. In fact I. M. James and E. Thomas, in a forthcoming paper, show that any vector bundle over an odd dimensional projective space which is s-isomorphic to the tangent bundle and has the same dimension must actually be isomorphic to the tangent bundle.

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