RESEARCH ANNOUNCEMENTS

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GROUPS WITH A BRUHAT DECOMPOSITION

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Introduction. In §1, we consider a group G which satisfies a simplified version of the axioms of Steinberg [**3**], and state some general results on the structure of G, the first two of which are due to J. Tits [**7**; **9**]. The main result is stated in §2, and can be viewed as a generalization of a theorem of Higman and McLaughlin [**2**, Theorem 2]. The theorem states essentially that a finite simple group G, which satisfies certain structural assumptions independent of the arithmetical structure of G, and whose Weyl group is isomorphic to the Weyl group of a complex simple Lie algebra $\mathfrak{g}$ of type A_n $(n \geqq 2)$, D_n $(n \geqq 4)$, or E_n $(n=6, 7, 8)$, is isomorphic to the group of Chevalley [**1**] determined by $\mathfrak{g}$ and some finite field K. The possibility of such a theorem can be seen in the paper of Tits [**8**], who showed that a group with a root data is the amalgamated product of the canonically imbedded subgroups of rank two. For a group G satisfying the hypotheses of the theorem, the subgroups of rank two and their amalgamation are uniquely determined by the Weyl group. The applicability of the theorem is limited to the groups associated with simple Lie algebras of types A_n $(n \geqq 2)$, D_n, and E_n because these are the simple groups of Chevalley all of whose canonically imbedded simple subgroups of rank two are of type A_2, and so can be classified by the result of Higman and McLaughlin.

1. **The structure of groups with a Bruhat decomposition.** Throughout this note, we shall use the following notations: $|A|$, cardinality of A; $A \lhd B$, A is normal in B; $\langle A, B, \cdots \rangle$, group generated by $A, B, \cdots$; $C(A)$, center of A; (A, B), group generated by all commutators $(a, b) = aba^{-1}b^{-1}$, $a \in A$, $b \in B$.

The groups considered in §1 are not assumed to be finite.

[1] This paper was written while the author was a Senior Postdoctoral Fellow of the National Science Foundation. The author expresses his warmest thanks to J. Tits for several stimulating and instructive conversations on this subject.

(1.1) DEFINITION. A group G will be called a *group with a Bruhat decomposition* if it satisfies the following conditions.

(i) There exist subgroups X, H of G such that $X\cap H=\{1\}$, XH is a group, and $X\lhd XH$.

(ii) There exists a subgroup W^* containing H such that $H\lhd W^*$, and $W^*\cap XH=H$.

(iii) The group $W=W^*/H$ contains elements $\{w_1, \cdots, w_n\}$ such that $w_i^2=1$, $i=1, \cdots, n$, and $W=\langle w_1, \cdots, w_n\rangle$.

Let $\{\omega(w), w\in W\}$ be a set of coset representatives of H in W^*, so that if $\zeta: W^*\to W$ is the natural mapping, then $\zeta(\omega(w))=w$ for all $w\in W$.

(iv) For some $\omega_0\in W^*$, $\omega_0 X\omega_0^{-1}\cap XH=\{1\}$.

We shall write ω_i for $\omega(w_i)$, $1\leqq i\leqq n$, and w_0 for $\zeta(\omega_0)$.

(v) For each i, $1\leqq i\leqq n$, there is a subgroup $X_i\neq\{1\}$ of X such that $X=X'_{w_i}X_i$, where $X'_{w_i}=\{x\in X: \omega_i x\omega_i^{-1}\in X\}$.

(vi) If, for $w\in W$, we set $X'_w=\{x\in X: \omega(w)x\omega(w)^{-1}\in X\}$, then for all i, $1\leqq i\leqq n$, and $w\in W$, either $X_i\subset X'_w$ or $X_i\subset X'_{ww_i}$.

(vii) For each i, $1\leqq i\leqq n$, $X_iH\cup X_iH\omega_iX_i$ is a subgroup of G.

(viii) $G=\langle X, W^*\rangle$.

These axioms are satisfied by any group possessing a root data in the sense of Tits [**8**], and by any group for which the axioms of Steinberg [**3**] hold (see [**3**, §4], [**4**, §14], and [**5**, §12]).

The structure of a group G with a Bruhat decomposition can be derived from the fact that the subgroups $B=XH$, $N=W^*$ form a (B, N)-pair in the sense of Tits [**9**], and is described by the following two results of Tits [**7**; **9**].

(1.1) $G=BW^*B=\bigcup_{w\in W} B\omega(w)B$; *moreover* $B\omega(w)B=B\omega(w')B$ *only if* $w=w'$.[2]

(1.2) *Let G be a group with a Bruhat decomposition such that* (a) X *is solvable*; (b) *it is impossible to split up the set* $\{w_1, \cdots, w_n\}$ *into two disjoint nonempty subsets which centralize each other*; *and* (c) *the subgroup G_1 of G generated by all conjugates of X is its own derived group. Then $C(G_1)\subset H$, and $G_1/C(G_1)$ is a simple group.*

The group W is called the *Weyl group* of G, and the elements $\{w_1, \cdots, w_n\}$ the *distinguished generators* of W. From the axioms (i)–(viii) it can be shown that the Weyl group of an arbitrary group with a Bruhat decomposition governs the behavior of the W^*-conjugates of $\{X_1, \cdots, X_n\}$. More precisely, we have

[2] This result generalizes the double coset decomposition for semi-simple Lie groups discovered by Bruhat, whose importance for abstract groups of Lie type was shown by Chevalley [**1**].

(1.3) *Let G and $\overline{G}$ be groups with Bruhat decompositions. Suppose there exists an isomorphism $w \to \bar{w}$ of the Weyl group W of G onto the Weyl group $\overline{W}$ of $\overline{G}$ which carries the distinguished generator w_i of W onto the distinguished generator $\bar{w}_i$ of $\overline{W}$. Let $\overline{X}$, $\overline{X}_i$, $\overline{X}'_{\bar{w}}$, etc. be the counterparts in $\overline{G}$ of X, X_i, X'_w, etc., in G. Then for all i, $1 \leqq i \leqq n$, and $w \in W$, $X_i \subset X'_w$ if and only if $\overline{X}_i \subset \overline{X}'_{\bar{w}}$. Moreover the mapping $\omega(w)X_i\omega(w)^{-1} \to \omega(\bar{w})\overline{X}_i\omega(\bar{w})^{-1}$ is a one-to-one mapping of the set of all W^*-conjugates of $\{X_1, \cdots, X_n\}$ onto the set of all $\overline{W}^*$-conjugates of $\{\overline{X}_1, \cdots, \overline{X}_n\}$.*

2. **On the determination of groups with Bruhat decompositions by their Weyl groups.** The main result is the following theorem.

(2.1) THEOREM. *Let G be a finite group with a Bruhat decomposition such that X is solvable, $X'_{w_i} \triangleleft X$ for $1 \leqq i \leqq n$, and each $G_i = \langle X_i, \omega_i X_i \omega_i^{-1}\rangle$ coincides with its derived group. Let there exist an isomorphism of the Weyl group of G onto the Weyl group of a complex simple Lie algebra $\mathfrak{g}$ of type A_n $(n \geqq 2)$, D_n $(n \geqq 4)$, or E_n $(n = 6, 7, 8)$ which carries distinguished generators onto distinguished generators. If for some i, $|X_i| > 4$, then there exists a finite field K such that the subgroup of G generated by the W^*-conjugates of $\{X_1, \cdots, X_n\}$ is a homomorphic image of the universal covering group Δ of the Chevalley group defined by $\mathfrak{g}$ and the field K (see Steinberg [6]). In particular, if G is generated by the W^*-conjugates of $\{X_1, \cdots, X_n\}$, then $G/C(G)$ is isomorphic to the simple group of Chevalley [1] determined by $\mathfrak{g}$ and K.*

We shall give a brief sketch of the proof. Full details will appear elsewhere. Let Σ be the system of roots of $\mathfrak{g}$ relative to a Cartan decomposition, and let $F = \{a_1, \cdots, a_n\}$ be a fundamental set of roots. We shall identify the Weyl group of G with the Weyl group of $\mathfrak{g}$, and may identify the distinguished generators $\{w_1, \cdots, w_n\}$ with the reflections w_{a_i} determined by the elements of F. Let G' be the Chevalley group associated with $\mathfrak{g}$ and an arbitrary field K. Then G' has a Bruhat decomposition with $\mathfrak{U}$, $\mathfrak{H}'$, $\mathfrak{W}' = G' \cap \mathfrak{W}$, $\mathfrak{X}_{a_i}$ (see [1]) taking the place of X, H, W^*, and X_i. By (1.3), there is a one-to-one correspondence between the set of W^*-conjugates of $\{X_1, \cdots, X_n\}$ and the $\mathfrak{W}'$-conjugates of $\{\mathfrak{X}_{a_1}, \cdots, \mathfrak{X}_{a_n}\}$. Since the latter are the one-parameter groups $\{\mathfrak{X}_r, r \in \Sigma\}$, we can define, for $r \in \Sigma$,

$$(2.2) \qquad X_r = \omega(w)X_i\omega(w)^{-1}, \qquad \text{if } w(a_i) = r.$$

Then by (1.3), X_r is defined in an unambiguous way, and $X_r \subset X$ if and only if $r > 0$.

(2.3) *Let r, s be independent roots of Σ, and let $G_r = \langle X_r, X_{-r}\rangle$, $r \in \Sigma$, and $G_{rs} = \langle G_r, G_s\rangle$. If the reflections w_r and w_s commute, then $(G_r, G_s) = 1$.*

If $(w_r, w_s) \neq 1$, *then* G_{rs} *is a group with a Bruhat decomposition, whose Weyl group is isomorphic to the symmetric group* S_3. *Moreover* G_{rs} *satisfies the conditions* (a)–(j) *of* [2, *p.* 395], *and by* (1.2) *and Theorem* 2 *of* [2], $G_{rs}/C(G_{rs})$ *is isomorphic to the Chevalley group of type* A_2 *defined over some finite field* K.

It can then be shown that the homomorphism of G_{rs} onto the Chevalley group of type A_2 can be chosen in such a way that X_r is mapped isomorphically onto a one-parameter group. Therefore $|X_r| = |K|$. By (2.2) and the properties of Σ, it follows that all subgroups $\{X_r, r \in \Sigma\}$ are conjugate in G, so that the field K is determined independently of the choice of the roots r, s such that $(w_r, w_s) \neq 1$.

Finally, it can be proved by an inductive argument that for each $r \in \Sigma$, there exists an isomorphism $t \to v_r(t)$ of the additive group of K onto X_r such that for all $r, s \in \Sigma$, $r+s \neq 0$,

$$(2.4) \qquad (v_r(t), v_s(t')) = v_{r+s}(N_{rs}tt'), \qquad t, t' \in K,$$

where $\{N_{rs}, r, s \in \Sigma\}$ are the normalized structure constants of $\mathfrak{g}$. Theorem (2.1) follows from (2.4) and the definition of the covering group Δ of Steinberg [6].

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