symmetric space V (irreducible or not) is again Hermitian symmetric and is isomorphic to V.

PROOF. Since $H^1(V, \theta)=0$ [2], we see that the set of points $t \in B$ for which V_t is isomorphic to V is an open set in B [3]; it is also closed by Theorem 2.

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ON THE BEST APPROXIMATION FOR SINGULAR INTEGRALS BY LAPLACE-TRANSFORM METHODS

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1. **Introduction.** Let $f(t)$ be a Lebesgue-integrable function in $(0, R)$ for every positive R. We denote by

$$J_\rho(t) = \rho \int_0^t f(t-u)k(\rho u)du \qquad (t \geqq 0)$$

a general singular integral with parameter $\rho > 0$ and kernel k having the following property (P): $k(u) \geqq 0$ in $0 \leqq u < \infty$, $k \in L(0, \infty)$, and $\int_0^\infty k(u)du = 1$.

If we restrict the class of functions $f(t)$ such that $e^{-ct}f \in L_p(0, \infty)$, $1 \leqq p < \infty$, for every $c > 0$, and if k satisfies (P), then the following statements hold:

(i) $J_\rho(t)$ exists as a function of t almost everywhere, $e^{-ct}J_\rho \in L_p(0, \infty)$ for every $c > 0$, and $\|e^{-ct}J_\rho\|_{L_p(0,\infty)} \leqq \|e^{-ct}f\|_{L_p(0,\infty)}$;

(ii) $\lim_{\rho \uparrow \infty} \|e^{-ct}\{f - J_\rho\}\|_p = 0$.

Furthermore, we denote by

$$\hat{f}(s) = \int_0^\infty e^{-st}f(t)dt \qquad (s = \sigma + i\tau, \ \mathrm{Re}\, s > 0)$$

the Laplace-transformation of a function f belonging to one of the classes described above, and the Laplace-Stieltjes-transform of a function $h(t)$ locally of bounded variation at each $t \geqq 0$ with $\int_0^\infty e^{-ct}|dh(t)| < \infty$ for every $c>0$ by

$$\check{h}(s) = \int_0^\infty e^{-st}dh(t) \qquad (\text{Re } s > 0).$$

Some fundamental hypotheses upon the kernel k are needed to prove the approximation theorems stated below. Let $\hat{k}(s)$ (Re $s \geqq 0$) be the Laplace-transform of k: At first,

$$(1.1) \qquad \lim_{\rho \uparrow \infty} (s/\rho)^{-\gamma}[1 - \hat{k}(s/\rho)] = A \qquad (\text{Re } s > 0)$$

should exist for some real $0<\gamma\leqq 1$, where A is a positive finite constant; secondly, there exists a normalized function $Q(u)$ of bounded variation in $[0, \infty]$ with $Q(\infty)=1$ such that

$$(1.2) \qquad A^{-1}(s/\rho)^{-\gamma}[1 - \hat{k}(s/\rho)] = \check{Q}(s/\rho) \qquad (\text{Re } s \geqq 0);$$

and thirdly, let there be a $q \in L(0, \infty)$, $\int_0^\infty q(u)du = 1$, and

$$(1.3) \qquad A^{-1}(s/\rho)^{-\gamma}[1 - \hat{k}(s/\rho)] = \hat{q}(s/\rho) \qquad (\text{Re } s \geqq 0).$$

It may be mentioned here that the conditions (1.2) and (1.3), respectively, imply (1.1), but the inverse does not seem to hold. Moreover, if the kernel k is not positive, then the constant γ need not be bounded by one.

2. Approximation theorems.

THEOREM 1. *Let $e^{-ct}f$, $e^{-ct}l \in L(0, \infty)$ for every $c>0$, let k satisfy* (*P*), *and let* (1.1) *hold for some real γ* ($0<\gamma\leqq 1$).

(i) *Then $\|e^{-ct}\{\rho^\gamma(f-J_\rho)-l\}\|_{L_1(0,\infty)} = o(1)$ $(\rho \uparrow \infty)$ implies*

$$As^\gamma\hat{f}(s) = \hat{l}(s) \qquad (\text{Re } s > 0)$$

or

$$f(t) = \frac{1}{A}\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}\, l(u)du \quad a.e.$$

(ii) *If $\|e^{-ct}\{f-J_\rho\}\|_{L_1(0,\infty)} = O(\rho^{-\gamma})$ $(\rho\uparrow\infty)$, then there exists a function $F(t)$ locally of bounded variation at $t \geqq 0$ with $\int_0^\infty e^{-ct}|dF(t)| < \infty$ for every $c>0$ such that*

$$(2.1) \qquad As^\gamma\hat{f}(s) = \check{F}(s) \qquad (\text{Re } s > 0).$$

SKETCH OF PROOF. As (i) can readily be shown, we will restrict ourselves to the proof of (ii). Clearly,

$$[1 - k(s/\rho)]\hat{f}(s) = \int_0^\infty e^{-st}\{f(t) - J_\rho(t)\}\,dt \qquad (\operatorname{Re} s > 0),$$

and if we define

$$S_T(t) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(1 - \frac{|\tau|}{T}\right)e^{st}[1 - \hat{k}(s/\rho)]\hat{f}(s)\,ds$$

$$(s = c + i\tau,\ c > 0),$$

then with the aid of the above equation it can be rewritten as

$$S_T(t) = \frac{2}{\pi T}\int_0^\infty e^{c(t-u)}\,\frac{\sin^2\{T(t-u)/2\}}{(t-u)^2}\,\{f(u) - J_\rho(u)\}\,du.$$

Now, the large O-approximation of f by J_ρ gives

$$\|e^{-ct}S_T\|_{L_1(-\infty,\infty)} \leqq \|e^{-ct}\{f - J_\rho\}\|_{L_1(0,\infty)} = O(\rho^{-\gamma}) \qquad (\rho \uparrow \infty)$$

for all $T \geqq 0$, and using the condition (1.1) and Lebesgue's dominated convergence theorem we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(1 - \frac{|\tau|}{T}\right)e^{st}As^\gamma\hat{f}(s)\,ds = \lim_{\rho\uparrow\infty}\rho^\gamma S_T(t),$$

finally, with Fatou's lemma

$$\left\|e^{-ct}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(1 - \frac{|\tau|}{T}\right)e^{st}As^\gamma\hat{f}(s)\,ds\right\|_{L_1(-\infty,\infty)}$$

$$\leqq \liminf_{\rho\uparrow\infty}\rho^\gamma\|e^{-ct}S_T\|_{L_1(-\infty,\infty)} = O(1)$$

for all $T \geqq 0$. Evidently the assumptions of the theorem and (1.1) give that $|As^\gamma\hat{f}(s)|$ is uniformly bounded in $\operatorname{Re} s \geqq \delta > 0$. Now, using a representation theorem for Laplace-transforms [1], this implies the existence of a function $F(t)$ such that (2.1) is valid.

The condition (2.1) defines a certain class K of functions f, and Theorem 1 shows, if there exists a constant γ such that (1.1) holds, and if $\|e^{-ct}\{f - J_\rho\}\|_{L_1(0,\infty)} = O(\rho^{-\gamma})$, then $f \in K$. The next theorem now shows that the inverse holds too.

THEOREM 2. *Let* $e^{-ct}f \in L(0, \infty)$ *for every* $c > 0$, *let* k *satisfy* (*P*), *and let the relation* (1.2) *be satisfied for some* $0 < \gamma \leqq 1$. *Then the assumption* (2.1) *implies* $\|e^{-ct}\{f - J_\rho\}\|_{L_1(0,\infty)} = O(\rho^{-\gamma})$ $(\rho \uparrow \infty)$.

If the two foregoing theorems hold, we say the singular integral J_ρ is said to be saturated with order $O(\rho^{-\gamma})$, and the functions f yielding

(2.1) define the saturation class of J_ρ. It was J. Favard [**5**] who introduced this terminology in approximation theory, and one of the authors [**2**; **3**] first made use of Fourier-transform methods to determine the saturation classes of singular integrals which are convolution integrals connected with the Fourier-transform. This question was independently but a little later treated by G. Sunouchi [**6**] too. Now in this paper general singular integrals are discussed which are classical convolution integrals connected with the Laplace-transform. Although there are connections between the Fourier- and Laplace-transform methods, it may be mentioned that the special properties and peculiar structure of the Laplace-transform play an important role in the proofs and the formulations of the stated theorems.

In the space $L_p(0, \infty)$, $1<p<\infty$, an equivalent theorem holds too.

THEOREM 3. *Let k satisfy* (*P*), *and let the condition* (1.3) *exist for some constant* γ $(0<\gamma\leqq 1)$. *A necessary and sufficient condition that the singular integral* $J_\rho(t)=\rho\int_0^t f(t-u)k(\rho u)du\,(\rho>0)$ *shall be saturated with order* $O(\rho^{-\gamma})$ *for functions* $e^{-ct}f\in L_p(0, \infty)$, $1<p<\infty$, $c>0$, *is that there exists a function* $e^{-ct}F\in L_p(0, \infty)$, $c>0$, *such that*

$$As^\gamma \hat{f}(s) = \hat{F}(s) \qquad (\mathrm{Re}\, s > 0)$$

or

$$f(t) = \frac{1}{A}\int_0^t \frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)} F(u)du \quad a.e.$$

3. **Application.** As an application we will consider a boundary value problem of heat conduction of a semi-infinite rod $(x\geqq 0)$. $U(x, t)$ is the temperature in the rod at time $t>0$, which is described by the equations

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U(x,t)}{\partial x^2} \ (x, t>0); \qquad \lim_{x\downarrow 0} U(x,t) = U_0(t) \qquad (t>0).$$

Among others, G. Doetsch [**4**, Bd. III] has shown that the solution is given by

$$U(x,t) = \frac{x}{2\sqrt{\pi}}\int_0^t U_0(t-u)\,\frac{\exp(-x^2/4u)}{u^{3/2}}\,du \qquad (x, t>0),$$

where U_0 is a Lebesgue-integrable function, and that the solution is unique, if the convergence of $U(x, t)$ to $U_0(t)$ is defined by the norm-convergence of the given function space.

$U(x, t)$ is a singular convolution integral with parameter $\rho=1/x^2>0$ and kernel

$$k(u) = \frac{1}{2\sqrt{\pi}} \frac{\exp(-1/4u)}{u^{3/2}} \qquad (0 \leqq u \leqq \infty).$$

It is easy to see that the kernel k has property (P) and its Laplace-transform $\hat{k}(s) = \exp(-\sqrt{s})$ satisfies the conditions (1.1), (1.2), and (1.3) for $\gamma = 1/2$ with $A = 1$.

If we restrict the temperature at the origin $U_0(t)$ such that $e^{-ct}U_0 \in L_p(0, \infty)$, $1 < p < \infty$, $c > 0$, for instance, then making use of Theorem 3 we have:

(i) $\|e^{-ct}\{U_0(t) - U(x, t)\}\|_{L_p(0,\infty)} = o(x)$ $(c > 0, x \downarrow 0)$ implies $U_0(t) = 0$ a.e.;

(ii) $\|e^{-ct}\{U_0(t) - U(x, t)\}\|_{L_p(0,\infty)} = O(x)$ $(c > 0, x \downarrow 0)$ guarantees that the flux of heat at the boundary $W_0(t)$ exists a.e., $e^{-ct}W_0 \in L_p(0, \infty)$ for every $c > 0$, and

$$\sqrt{s}\hat{U}_0(s) = \hat{W}_0(s) \qquad (\operatorname{Re} s > 0)$$

or, equivalently,

$$U_0(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{W_0(u)}{\sqrt{t - u}} du \quad \text{a.e.},$$

and vice versa.

The complete proofs of these and further results as well as a detailed discussion will appear elsewhere (see [7]).

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