DEFORMATIONS OF RIEMANNIAN STRUCTURES

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It has been shown in [1] that a Kählerian deformation of an irreducible compact Hermitian symmetric space V is again symmetric and is isomorphic to it. The aim of this note is to prove this for any compact Hermitian symmetric space, irreducible or not. We first prove the following Theorem 1 which is a slight generalization[1] of Theorem 2 in [1].

THEOREM 1. *Let V and B be two differentiable manifolds and let $R(t)$ be a riemannian structure on V which depends in a differentiable way on $t \in B$; let F be any compact riemannian manifold. Then the set of points $t \in B$ such that $(V, R(t))$ is isomorphic to F is closed in the set of points $t \in B$ such that $(V, R(t))$ is complete.*

PROOF. Denote by C the set of all points $t \in B$ such that $V_t = (V, R(t))$ is complete and let t_n be a sequence of points in C converging to a point t_0 such that (i) $t_0 \in C$, and (ii) $V_{t_n} = (V, R(t_n))$ is isomorphic to F for all $n > 0$. We prove that V_0 is isomorphic to F. Let h_n be an isomorphism of F onto $V_{t_n} = (V, R(t_n))$ and let r_0 be an orthonormal frame in the tangent space $T_0(F)$ at p_0 of F; denote by r_{t_n} the orthonormal frame $h_n(r_0)$ at $x_n = h_n(p_0)$ in V_{t_n}. Since V_0 is compact, the sequence of points $\{x_n\}$ has a limit point x_0. Since the set of all orthonormal frames at all points of a compact neighbourhood of x_0 in V_0 is again compact, the sequence $\{r_{t_n}\}$ admits a limit point r_0'. Let l_t be the linear mapping of $T_{x_1}(V_t)$ onto $T_{x_n}(V_{t_n})$ which maps r_1 onto r_{t_n}; let u be a unit vector at x_1 and let y be the end-point of the geodesic arc of length s, of origin x_1 and tangent to u; then $y = \phi(s, u)$ where ϕ is a differentiable function of u and of s. If y_t is the end-point of the geodesic arc of origin x_t and tangent to $l_t(u)$, we have $y_t = \phi(s, u, t)$ where $\phi(s, u, 1) = \phi(s, u)$. For $t > 0$, we have $y_t = h_t(y)$; as $t_n \to 0$, $h_n(u)$ tends to a vector $h_0(u)$ and $h_n(y)$ tends to a point $h_0(y) = \phi(s, u, 0)$. It is easy to see that h_0 is a well-defined differentiable mapping of V_1 into V_0; since V_0 is complete, it follows that h_0 is onto. We prove that h_0 is one-to-one; let z_1 and z_2 be two distinct points of V_1 and let d_t denote the metric defined by $R(t)$ on V_t. Since

[1] This generalization of Theorem 2 in [1] has been suggested to me by Professor J. L. Koszul. The proof of Theorem 2 [1] has been suggested to me by Professor C. Ehresmann. This research is supported in part by NSF G-18834.

h_t is an isomorphism of F onto V_t, we have $d_t(h_t(z_1), h_t(z_2)) = d_1(z_1, z_2)$. Suppose that $h_0(z_1) = h_0(z_2)$; we have

$$d_t(h_t(z_1), h_t(z_2)) < d_t(h_0(z_1), h_t(z_1)) + d_t(h_0(z_2), h_t(z_2)).$$

Since $h_t(z_1)$ and $h_t(z_2)$ tend to $h_0(z_1)$ as $t \to 0$, we see that $d_t(h_t(z_1), h_t(z_2)) = d_1(z_1, z_2) = 0$, a contradiction. Thus to prove that h_0 is an isomorphism, it is sufficient to prove that h_0 is a local isomorphism. Let U_1 be a normal geodesic neighbourhood of x_1 in V_1; for each $z \in U_1$, there exists a unique geodesic γ_1 of origin x_1 and end-point $z = \phi(a, u)$. h_0 associates to each geodesic γ_1 ni U_1 a geodesic γ_0 of origin x_0 and to each normal coordinate system in U_1 a normal coordinate system in a neighbourhood U_0 of x_0 in V_0; hence the restriction of h_0 to U_1 is an isomorphism. Since γ_1 is compact, there exists a positive number ϵ such that there exists a geodesic coordinate system of center $z = \phi(a, u)$ in V_1 and of center $\phi(s, u, 0) = h_0(z)$ of radius ϵ for $0 < s < a$. If h_0 is a local isomorphism in the neighbourhood of z in V_1, the restriction of h_0 to a ball of radius $< \epsilon$ of center z is an isomorphism; in fact, let r_1' be an orthonormal frame at z in V_1 and let $h_0(r_1')$ be the corresponding frame in V_0; by the corresponding geodesic arcs of origins z and $h_0(z)$, we can define as above, a differentiable mapping h_0 of V_1 onto V_0 and it is easy to see that $h_0' = h_0$. Let s_1 be the upper bound of the values of s for which h_0 is a local isomorphism of a neighbourhood of $\phi(s, u)$ in V_1 into V_0; then we can find an s' such that there exists a geodesic ball of center $\phi(s', u)$ which contains $\phi(s_1, u)$; by repeating the above argument, one can show that h_0 is a local isomorphism in a neighbourhood of $\phi(s_1, u)$, which leads to a contradiction. Hence h_0 is a local isomorphism in the neighbourhood of each point and the theorem follows.

A similar argument proves the following:

THEOREM 2. *Let V be a complex manifold and let $K(t)$ be a Kähler metric on V depending in a differentiable way on a parameter $t \in B$, where B is a differentiable manifold; let F be a compact Kähler manifold. Then the set of points $t \in B$ for which $(V, K(t))$ is isomorphic to F is closed in the set of points $t \in B$ such that $(V, K(t))$ is complete.*

PROOF. We may assume that h_t is an isomorphism (complex analytic) of F onto V_t and consider the unitary frames in the proof of the above theorem; then the mapping $\phi(x_1, u)$ is complex analytic. It follows, then, that h_0 is a complex analytic mapping of V_1 onto V_0 and it can be shown, as above, that h_0 is a local isomorphism.

As an application, we have the following:

COROLLARY. *Any Kählerian deformation of a compact Hermitian* *symmetric space V (irreducible or not) is again Hermitian symmetric and is isomorphic to V.*

PROOF. Since $H^1(V, \theta)=0$ [2], we see that the set of points $t\in B$ for which V_t is isomorphic to V is an open set in B [3]; it is also closed by Theorem 2.

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ON THE BEST APPROXIMATION FOR SINGULAR INTEGRALS BY LAPLACE-TRANSFORM METHODS

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1. **Introduction.** Let $f(t)$ be a Lebesgue-integrable function in $(0, R)$ for every positive R. We denote by

$$J_\rho(t) = \rho \int_0^t f(t-u)k(\rho u)du \qquad (t \geqq 0)$$

a general singular integral with parameter $\rho>0$ and kernel k having the following property (P): $k(u)\geqq 0$ in $0\leqq u<\infty$, $k\in L(0,\infty)$, and $\int_0^\infty k(u)du=1$.

If we restrict the class of functions $f(t)$ such that $e^{-ct}f\in L_p(0,\infty)$, $1\leqq p<\infty$, for every $c>0$, and if k satisfies (P), then the following statements hold:

(i) $J_\rho(t)$ exists as a function of t almost everywhere, $e^{-ct}J_\rho \in L_p(0,\infty)$ for every $c>0$, and $\|e^{-ct}J_\rho\|_{L_p(0,\infty)}\leqq \|e^{-ct}f\|_{L_p(0,\infty)}$;

(ii) $\lim_{\rho\uparrow\infty}\|e^{-ct}\{f-J_\rho\}\|_p=0$.

Furthermore, we denote by

$$\hat{f}(s) = \int_0^\infty e^{-st}f(t)dt \qquad (s=\sigma+i\tau, \operatorname{Re} s>0)$$

the Laplace-transformation of a function f belonging to one of the classes described above, and the Laplace-Stieltjes-transform of a