THE DOMAIN OF DEPENDENCE INEQUALITY FOR SYMMETRIC HYPERBOLIC SYSTEMS[1]

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1. **Introduction.** Energy propagates through space with finite speed. This is a characteristic feature of wave propagation phenomena and, indeed, it is a postulate in relativistic physics. Recently [3; 4] the author demonstrated this property by means of *a priori* "domain of dependence" inequalities for solutions of initial-boundary value problems for Maxwell's equations and second order hyperbolic equations and used the inequalities to prove existence, uniqueness and regularity theorems for these problems. In this announcement the results are extended to the most general systems of linear partial differential equations that admit an energy integral, the symmetric hyperbolic systems of K. O. Friedrichs [1].

The systems have the form

$$(1.1) \quad E_{\alpha\beta}(x) \frac{\partial u_\beta}{\partial t} = A^i_{\alpha\beta}(x) \frac{\partial u_\beta}{\partial x_i} + B_{\alpha\beta}(x) u_\beta + f_\alpha(x, t) \quad (\alpha = 1, 2, \cdots, m),$$

where $E_{\alpha\beta}$, $A^t_{\alpha\beta}$ and $B_{\alpha\beta}$ are real-valued functions of $x = (x_1, x_2, \cdots, x_n) \in R^n$ and $u_\beta = u_\beta(x, t)$ and f_α are real-valued functions of x and $t \in R^1$. The summation convention is used; i.e., repeated indices i and β are summed over their ranges $(1 \leq i \leq n, 1 \leq \beta \leq m)$. System (1.1) may also be written in matrix notation

$$E \frac{\partial u}{\partial t} = A^i \frac{\partial u}{\partial x_i} + Bu + f,$$

where $E = (E_{\alpha\beta})$, $A^i = (A^t_{\alpha\beta})$ and $B = (B_{\alpha\beta})$ are $m \times m$ matrices and $u = (u_\alpha)$ and $f = (f_\alpha)$ are $m \times 1$ (column) matrices.

A system (1.1) is symmetric hyperbolic if E and A^i $(i = 1, 2, \cdots, n)$ are symmetric and E is positive definite. The form $\eta = \frac{1}{2} E_{\alpha\beta} u_\alpha u_\beta$ is interpreted as an "energy density" (energy per unit volume). The forms $\Sigma_i = -\frac{1}{2} A^t_{\alpha\beta} u_\alpha u_\beta$ are interpreted as the components of a "Poynting vector" describing the flow of power (energy per unit area per unit time). They are related by the "conservation of energy law"

$$\frac{\partial \eta}{\partial t} + \frac{\partial \Sigma_i}{\partial x_i} = \frac{1}{2} \left(B_{\alpha\beta} + B_{\beta\alpha} - \frac{\partial A^i_{\alpha\beta}}{\partial x_i} \right) u_\alpha u_\beta + f_\alpha u_\alpha.$$

The energy per unit time entering a domain $\Omega \subset R^n$ through its boundary $\partial\Omega$ is

(1.2) $\int_{\partial\Omega} \Sigma_i n_i dS = \int_{\partial\Omega} \eta v_i n_i dS$ (n_i = normal vector on $\partial\Omega$)

where $v_i = \Sigma_i/\eta$. Thus, v_i may be interpreted as a "velocity" of energy flow, which suggests that the speed with which energy propagates in Ω is bounded above by

(1.3) $$c = \sup \frac{-A^i_{\alpha\beta}(x)\xi_\alpha\xi_\beta n_i}{E_{\alpha\beta}(x)\xi_\alpha\xi_\beta},$$

where the supremum is over $x \in \Omega$, $\xi \in R^m$ and unit vectors n_i ($n_i n_i = 1$). This property of c is verified below (§3) by the domain of dependence inequality.

This announcement deals with initial-boundary value problems for (1.1) in domains Ω, subject to dissipative, local boundary conditions on $\partial\Omega$. Precise definitions of the boundary conditions are given in §2. The domain of dependence inequality is formulated and proved in §3. Its application to existence, uniqueness, and regularity theorems for initial-boundary value problems, together with complete proofs of all the results and their application to particular systems, will be given elsewhere (see [5]).

Throughout this paper, Ω denotes an arbitrary domain (open connected set) in R^n, and $E_{\alpha\beta}(x)$, $A^i_{\alpha\beta}(x)$ and $B_{\alpha\beta}(x)$ are assumed to be defined (almost everywhere) in Ω and to satisfy the conditions

(1.4) $E_{\alpha\beta}(x) = E_{\beta\alpha}(x)$, $A^i_{\alpha\beta}(x) = A^i_{\beta\alpha}(x)$, $B_{\alpha\beta}(x)$ and $\partial A^i_{\alpha\beta}(x)/\partial x_i$

(which is defined as a distribution) are bounded, Lebesgue-measurable functions in Ω, and

(1.5)   There exists a positive constant E^0 such that

$E_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geqq E^0\xi_\alpha\xi_\alpha$ for $x \in \Omega$ and $\xi \in R^m$.

2. **Boundary conditions.** If $\phi = (\phi_\alpha) \in C_0^1(\Omega)$ then

$$(A\phi)_\alpha = A^i_{\alpha\beta}\frac{\partial\phi_\beta}{\partial x_i} + \frac{1}{2}\frac{\partial A^i_{\alpha\beta}}{\partial x_i}\phi_\beta \qquad (\alpha = 1, 2, \cdots, m)$$

defines an operator $A: C_0^1(\Omega) \to L_2(\Omega)$, and A^+, the formal adjoint of A, equals $-A$. This suggests the

DEFINITION. If u is in $L_2(\Omega)$, then we say that Au exists and is

equal to v in $L_2(\Omega)$ provided that

$$\int_\Omega u_\alpha(A\phi)_\alpha dx + \int_\Omega v_\alpha\phi_\alpha dx = 0 \qquad \text{for all } \phi \in \overset{1}{C_0}(\Omega).$$

The space $L_2(A; \Omega) = \{u : u \text{ and } Au \text{ are in } L_2(\Omega)\}$ is a separable Hilbert space with respect to the scalar product

$$(u, v)_{L_2(A;\Omega)} = \int_\Omega \{u_\alpha v_\alpha + (Au)_\alpha(Av)_\alpha\} dx.$$

In classical theory, the set of functions u which satisfy a linear homogeneous boundary condition on $\partial\Omega$ and have $Au \in L_2(\Omega)$ form a linear space containing $C_0^1(\Omega)$. This suggests the

DEFINITION. A boundary condition Γ for A is a closed linear subspace Γ of $L_2(A; \Omega)$ such that $C_0^1(\Omega) \subset \Gamma$.

Each boundary condition Γ for A determines an "adjoint" boundary condition Γ^* for A defined by

$$\Gamma^* = L_2(A; \Omega) \cap \left\{v: \int_\Omega \{u_\alpha(Av)_\alpha + (Au)_\alpha v_\alpha\} dx = 0 \text{ for all } u \in \Gamma\right\}.$$

It follows that $(\Gamma^*)^* = \Gamma$. Thus, boundary conditions for A occur in adjoint pairs.

DEFINITION. Γ is a dissipative boundary condition for A provided that

$$\int_\Omega u_\alpha(Au)_\alpha dx \leqq 0 \qquad \text{for all } u \in \Gamma.$$

This concept is equivalent to that of a dissipative operator due to R. S. Phillips [2]; see [5].

DEFINITION. Γ is a local boundary condition for A provided that $\phi u \in \Gamma$ whenever $\phi \in C_0^1(R^n)$ and $u \in \Gamma$.

Examples of local, dissipative boundary conditions are provided by the classical systems of acoustics, electromagnetics and elasticity. With each is associated a pair of "natural" boundary conditions, corresponding to hard and soft boundaries in acoustics, perfect electrical and magnetic conductors in electromagnetics, and fixed and free boundaries in elasticity; see [5] where this is verified and a class of systems and boundary conditions generalizing these is described.

3. **The domain of dependence inequality.** Each boundary condition Γ for A is a closed subspace of $L_2(A; \Omega)$. Hence, Γ is a separable Hilbert space relative to the scalar product of $L_2(A;\Omega)$. If $I = \{t: 0 < t < T\}$ then the space

$$F = L_2(I; \Gamma) \cap \overset{1}{L_2}(I; L_2(\Omega))$$

(see [4, §2] for definitions) defines a class of functions $u = u(x, t)$ for which

$$E \frac{\partial u}{\partial t}, \quad A^i \frac{\partial u}{\partial x_i} = Au - \frac{1}{2} \frac{\partial A^i}{\partial x_i} u \text{ and } Bu$$

exist in $L_2(Q)$, $Q = \Omega \times I$, and u satisfies the boundary condition Γ ($u(t) \in \Gamma$ for almost every $t \in I$). The domain of dependence inequality is concerned with the related space F^{loc} for which $E(\partial u/\partial t)$, etc., are in $L_2^{\text{loc}}(Q)$. To describe it let

$$\Gamma^{\text{vox}} = \Gamma \cap \{u: u \text{ is equivalent to zero outside a bounded set } K \subset \Omega\},$$

and

$$\Gamma^{\text{loc}} = L_2^{\text{loc}}(A; \Omega)$$

$$\cap \left\{u: \int_\Omega \{u_\alpha (Av)_\alpha + (Au)_\alpha v_\alpha\} dx = 0 \text{ for all } v \in (\Gamma^*)^{\text{vox}}\right\}$$

and define

$$F^{\text{loc}} = \{u: u \in L_2(I; L_2(A; K \cap \Omega)) \cap L_2^1(I; L_2(K \cap \Omega)) \text{ for each bounded measurable set } K \subset R^n, \text{ and } u(t) \in \Gamma^{\text{loc}} \text{ for almost all } t \in I\}.$$

THEOREM (DOMAIN OF DEPENDENCE INEQUALITY). *Let Γ be a local, dissipative boundary condition for A. Then the speed c defined by (1.3) is finite and every $u \in F^{\text{loc}}$ satisfies*

$$\text{(3.1)} \quad \int_{\Omega \cap S(x^0, a)} E_{\alpha\beta} u_\alpha(T) u_\beta(T) dx$$

$$\leqq e^{kT} \left[\int_{\Omega \cap S(x^0, a+cT)} E_{\alpha\beta} u_\alpha(0) u_\beta(0) dx + 2 \int_{Q \cap C(x^0, a)} e^{-kt} f_\alpha u_\alpha dx dt\right]$$

where

$$f = E \frac{\partial u}{\partial t} - Au - \left(B - \frac{1}{2} \frac{\partial A^i}{\partial x_i}\right) u,$$

$$S(x^0, a) = \{x: |x - x^0| \leqq a\} \text{ and}$$

$$C(x^0, a) = \{(x, t): |x - x^0| \leqq a + c(T - t), 0 \leqq t \leqq T\}.$$

The constant k depends on n and the bounds for the coefficients only.

PROOF. The finiteness of c follows from (1.4) and (1.5). The proof is omitted. Also, only the special case $B = \frac{1}{2}\partial A^i/\partial x_i$ is treated. To prove (3.1), take $\delta > 0$ and let $\psi_\delta \in C_0^\infty(R^1)$ have the properties $\psi_\delta(\tau) \equiv 0$ for $\tau \geq \delta$, $\psi_\delta(\tau) \equiv 1$ for $\tau \leq -\delta$, and $\psi_\delta'(\tau) \leq 0$ everywhere (so $0 \leq \psi_\delta(\tau) \leq 1$ everywhere). Put $\phi(x, t) = \psi_\delta(\tau)$, $\tau = |x - x^0| - a - c(T - t)$. Then $\phi \in C_0^\infty(\bar{Q})$ for δ sufficiently small, and ϕ tends to the characteristic function of $C(x^0, a)$ when $\delta \to 0+$.

Let $u \in F^{\text{loc}}$, define $f \equiv E(\partial u/\partial t) - Au$ and consider

$$(3.2) \quad \int_Q f_\alpha u_\alpha \phi \, dx dt = \int_Q \phi u_\alpha E_{\alpha\beta} \frac{\partial u_\beta}{\partial t} \, dx dt - \int_Q \phi u_\alpha (Au)_\alpha dx dt = I_1 - I_2.$$

Integration by parts with respect to t gives [4, Theorem 2.3]

$$(3.3) \quad I_1 = \frac{1}{2} \int_\Omega \phi E_{\alpha\beta} u_\alpha u_\beta \Big]_0^T dx - \frac{1}{2} \int_Q (E_{\alpha\beta} u_\alpha u_\beta) \frac{\partial \phi}{\partial t} \, dx dt.$$

Next, $\sqrt{(\phi(\cdot, t))} \in C_0^1(R^n)$ for each $t \in I$, and $u(\cdot, t) \in \Gamma^{\text{loc}}$ for almost every $t \in I$. Hence, $\sqrt{(\phi(\cdot, t))}u(\cdot, t) \in \Gamma$ because Γ is a local boundary condition for A. Thus, since Γ is dissipative for A,

$$(3.4) \quad 0 \geq \int_\Omega \sqrt{(\phi(x, t))} u_\alpha(x, t) (A\sqrt{(\phi(x, t))} u(x, t))_\alpha dx$$

$$\text{for almost all } t \in I.$$

Now, $(A\psi u)_\alpha = \psi(Au)_\alpha + A_{\alpha\beta}^i(\partial\psi/\partial x_i)u_\beta$, whence (3.4) may be written

$$0 \geq \int_\Omega \phi(x, t) u_\alpha(x, t) (Au(x, t))_\alpha dx$$

$$+ \int_\Omega \sqrt{(\phi(x, t))} \frac{\partial\sqrt{(\phi(x, t))}}{\partial x_i} A_{\alpha\beta}^i u_\alpha(x, t) u_\beta(x, t) dx \quad (\text{almost all } t \in I).$$

Integrating over $t \in I$ and using

$$\sqrt{\phi} \frac{\partial\sqrt{\phi}}{\partial x_i} = \frac{1}{2} \frac{\partial\phi}{\partial x_i}$$

gives

$$(3.5) \quad 0 \geq I_2 + \frac{1}{2} \int_Q A_{\alpha\beta}^i u_\alpha u_\beta \frac{\partial\phi}{\partial x_i} \, dx dt.$$

Combining (3.5) and (3.3) with (3.2) gives

$$(3.6) \quad \int_Q f_\alpha u_\alpha \phi \, dx dt \geq \frac{1}{2} \int_\Omega E_{\alpha\beta} u_\alpha u_\beta \phi \Big]_0^T dx$$

$$- \frac{1}{2} \int_Q \left(E_{\alpha\beta} u_\alpha u_\beta \frac{\partial\phi}{\partial t} - A_{\alpha\beta}^i u_\alpha u_\beta \frac{\partial\phi}{\partial x_i} \right) dx dt.$$

But

$$\frac{\partial \phi}{\partial t} = \psi_\delta'(\tau)c,$$

$$\frac{\partial \phi}{\partial x_i} = \psi_\delta'(\tau)(x_i - x_i^0)/|x - x^0| = \psi_\delta'(\tau)n_i \quad \text{where} \quad n_i n_i = 1.$$

Hence, the integrand in the last integral is

(3.7) $$\psi_\delta'(\tau)(E_{\alpha\beta}u_\alpha u_\beta c - A^i_{\alpha\beta}u_\alpha u_\beta n_i) \leqq 0,$$

since $\psi_\delta'(\tau) \leqq 0$ and $E_{\alpha\beta}(x)\xi_\alpha\xi_\beta c - A^i_{\alpha\beta}(x)\xi_\alpha\xi_\beta n_i \geqq 0$ for all $x \in \Omega$, $\xi \in R^m$ and n_i with $n_i n_i = 1$, by the definition of c. Combining (3.6) and (3.7) gives the estimate

(3.8)
$$\int_Q f_\alpha u_\alpha \phi dx dt \geqq \frac{1}{2} \int_\Omega E_{\alpha\beta} u_\alpha(T) u_\beta(T) \phi(T) dx$$
$$- \frac{1}{2} \int_\Omega E_{\alpha\beta} u_\alpha(0) u_\beta(0) \phi(0) dx.$$

Finally, when $\delta \to 0$, ϕ tends boundedly to the characteristic function for $C(x^0, a)$. Hence, passage to the limit in (3.8) gives (3.1) with $k = 0$, which completes the proof for the case $B = \frac{1}{2}\partial A^i/\partial x_i$. In the general case, a simple variant of this argument gives (3.1) with a nonzero k (cf. [5, Theorem 6.1]).

References

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