WEIGHTED TRIGONOMETRICAL APPROXIMATION ON R^1 WITH APPLICATION TO THE GERM FIELD OF A STATIONARY GAUSSIAN PROCESS

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Given an even, nonnegative, Lebesgue measurable weight $\Delta = \Delta(a)$ $(a \in R^1)$ with $\int \Delta < \infty$, let Z be the (real) Hilbert space of Lebesgue measurable functions f with $f^*(-a) = f(a)$ and $\|f\| = \sqrt{\int |f|^2 \Delta} < \infty$, subject to the usual identifications, let Z^{cd} be the span (in Z) of e^{iat} $(c \leqq t \leqq d)$, and introduce the following subspaces of Z:

(a) $Z^- = Z^{-\infty 0}$,

(b) $Z^+ = Z^{0\infty}$,

(c) $Z^{+/-} =$ the projection of Z^+ upon Z^-,

(d) $Z^\bullet =$ the class of entire functions $f = f(\gamma)$ $(\gamma = a + ib)$ of minimal exponential type which, restricted to the line $b = 0$, belong to Z,

(e) $Z^{0+} = \bigcap_{\delta > 0} Z^{0\delta}$,

(f) $Z_\bullet =$ the span of (real) polynomials of ia belonging to Z,

(g) $Z^{-\infty} = \bigcap_{t<0} Z^{-\infty t}$.

Δ is a Hardy weight if

$$\int \frac{lg^- \Delta}{1 + a^2} > -\infty;$$

such a Hardy weight is expressible as $|h|^2$, h being an (outer) function belonging to the Hardy class of functions $f(\gamma)$ $(\gamma = a + ib)$ $(\gamma^* = a - ib)$ regular in the half plane $(b > 0)$ with $f^*(-a) = f(a)$ and $\int |f(a + ib)|^2 da$ bounded $(b > 0)$. $Z \neq Z^-$ or $Z = Z^- = Z^{-\infty}$ according as Δ is Hardy or not, a fact that goes back to Szegö.

Given a Hardy weight, it can be proved that

$$Z^- \supset Z^{+/-} \supset Z^- \cap Z^+ \supset Z^\bullet \supset Z^{0+} \supset Z_\bullet,$$

and the problem is to decide if some or all of the above subspaces coincide, special attention being paid to $Z^{+/-}$ and Z^{0+} for probabilistic reasons explained below. $Z^{0+} = Z^\bullet$ for the general Hardy weight, but the other inclusions can be strict; for instance, $Z^- \neq Z^{+/-}$ if and only if $\mathfrak{j} = h/h^*$, restricted to the line $b = 0$, agrees with the ratio of two inner functions, while $Z^{+/-} = Z^\bullet$ $(= Z^{0+})$ if and only if the reciprocal h^{-1} of the outer Hardy function h figuring in $\Delta = |h|^2$ is an entire function of minimal exponential type. $Z^\bullet \neq Z_\bullet$ is possible even for such

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nice weights. $Z^{\bullet}$ need not be a closed subspace of Z if Δ is non-Hardy, but $Z^{\bullet} \subset Z^{0+}$ for a general weight.

Δ can be viewed as the spectral weight of a (real) centered stationary Gaussian process with sample paths $t \rightarrow x(t)$, probabilities $P(B)$, and expectations $E(f)$:

$$E[x(s)x(t)] = \int e^{ia(t-s)}\Delta.$$

Now the map $x(t) \rightarrow e^{iat}$ effects an isomorphism between the (real) Hilbert space Q, obtained by closing up (real) combinations of $x(t)$ $(t \in R^1)$ under the norm $\|f\| = \sqrt{E(f^2)}$, and Z, and introducing the span Q^{cd} of $x(t)$ $(c \leqq t \leqq d)$ in Q and the corresponding field $\mathbf{B}^{cd}$, a perfect correspondence is obtained between

(a) Z^-, $Q^- = Q^{-\infty 0}$, and $\mathbf{B}^- = \mathbf{B}^{-\infty 0} =$ the past,

(b) Z^+, $Q^+ = Q^{0\infty}$, and $\mathbf{B}^+ = \mathbf{B}^{0\infty} =$ the future,

(c) $Z^{+/-}$, $Q^{+/-} =$ the projection of Q^+ upon Q^-, and $B^{+/-} =$ the minimal splitting field,

(d) Z^{0+}, $Q^{0+} = \bigcap_{\delta>0} Q^{0\delta}$, and $\mathbf{B}^{0+} = \bigcap_{\delta>0} \mathbf{B}^{0\delta} =$ the germ,

(e) $Z_{\bullet}$, $Q_{\bullet} =$ the span of such derivatives (in Q) as the sample path admits at $t=0$, and $\mathbf{B}_{\bullet} =$ the corresponding field,

(f) $Z^{-\infty}$, $Q^{-\infty} = \bigcap_{t<0} Q^{-\infty t}$, and $\mathbf{B}^{-\infty} = \bigcap_{t<0} \mathbf{B}^{-\infty t} =$ the distant past or tail field.

$\mathbf{B}^-$, $\mathbf{B}^+$, $\mathbf{B}^{+/-}$, etc. are not just the smallest fields over which Q^-, Q^+, $Q^{+/-}$, etc. are measurable, but if, for instance, $f \in Q$ is measurable over $\mathbf{B}^{+/-}$, then it belongs to $Q^{+/-}$.

$\mathbf{B}^{+/-}$, the minimal splitting field, needs some explanation.

Given a pair of fields $\mathbf{B}^-$ and $\mathbf{B}^+$ (such as the past and future described above), a field $\mathbf{B} \subset \mathbf{B}^-$ is a splitting field of $\mathbf{B}^-$ and $\mathbf{B}^+$ if, conditional on $\mathbf{B}$, $\mathbf{B}^+$ is independent of $\mathbf{B}^-$, $\mathbf{B}^-$ is itself a splitting field, the intersection of two splitting fields is again a splitting field, and a minimal splitting field exists, coinciding in the present (Gaussian) case with the field $\mathbf{B}^{+/-}$ of the projection of the future upon the past.

Given a Hardy weight $\Delta = |h|^2$, or what is the same, if the tail field $\mathbf{B}^{-\infty}$ is trivial, the motion splits over its germ ($\mathbf{B}^{+/-} = \mathbf{B}^{0+}$) if and only if $Z^{+/-} = Z^{0+}$, and according to the result announced above, that happens if and only if h^{-1} is an entire function of minimal exponential type; this is the principal probabilistic result of the investigation. To the best of our knowledge, the only published fact about $\mathbf{B}^{0+}$ is the lemma of Tutubalin-Freidlin [Teor. Veroyatnost. i Primenen. **1** (1961), 196–199] that if Δ is larger than the reciprocal of a polynomial near $\pm\infty$, then $\mathbf{B}^{0+} = \mathbf{B}_{\bullet}$.

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