RESEARCH ANNOUNCEMENTS

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ANALYTIC FUNCTIONS AND DIRICHLET PROBLEM[1]

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Communicated by Ralph Phillips, October 9, 1963

In a recent paper, K. Hoffman [6], brings to a considerable degree of generality a certain part of the theory of analytic functions which in the classical context deals with the Hardy classes H^p. He also discusses analytic structures in the maximal ideal spaces of corresponding function algebras. For the setting of these general results, related to earlier work of Arens and Singer [1; 2], Helson and Lowdenslager [5], Bochner [4], and others, the concept of a logmodular algebra is introduced, generalizing that of Dirichlet algebra, and most of the results are valid in the latter situation. This development leans heavily, as Hoffman points out, on two fundamental properties of logmodular algebras:

(i) The representing measure (see [6]) associated with a point of the maximal ideal space is uniquely determined.

(ii) If μ is a representing measure, A a logmodular algebra, then $A+\bar{A}$ is dense in $L^2(\mu)$.

The purpose of this abstract is to put in evidence the basic role played by (i) in the whole theory. Thus, for instance, it was not known whether conditions (i) and (ii) imposed on an arbitrary sup-norm algebra (see [6]) would be enough to recover in their full strength all the results valid for logmodular algebras. Actually, we shall show that (i)$\Rightarrow$(ii), and that the (additional) assumption of (i) alone yields all the results established in [6] for logmodular algebras. From a heuristic point of view this does not appear as an accident, since (i) is equivalent to the solvability of a certain Dirichlet problem associated with the algebra, to whose exact meaning we shall return below.

I. **Uniqueness of representing measures.** In what follows, A shall always denote a sup-norm algebra, in the sense of [6], on the compact Hausdorff space X. $\mathfrak{M}$ shall denote the maximal ideal space of A, $\hat{A}$ the image of A under the Gelfand representation; and for $m \in \mathfrak{M}$,

[1] This research was supported by National Science Foundation grant G24502.

$\hat{f}(m) = m(f)$ shall as usual denote the value of the homomorphism corresponding to m, at $f \in A$ [7]. We consider X imbedded in $\mathfrak{M}$ in the usual manner (thus homeomorphically), without further explanation, whenever needed below. Now, closer examination of the work mentioned above suggests that the fundamental features in the sup-norm algebras which are studied there are the following: $X \subset \mathfrak{M}$ is the Šilov boundary of $\hat{A}$, and fundamentally, that the Dirichlet problem can be solved, in some sense, in terms of $\operatorname{Re} A = \{\operatorname{Re} f : f \in A\}$ for each real-valued continuous function on X. Using the notions introduced by H. Bauer [3], we can reformulate this in the following precise manner:

(a) $X \subset \mathfrak{M}$ is the Šilov boundary of $\hat{A}$.

(b) Each continuous real-valued function on X is $\operatorname{Re} A$-resolutive in the sense of [3].

The following simple equivalent condition thus appears as providing a natural general setting for the theory considered:

THEOREM 1. *The conditions* (a) *and* (b) *hold if and only if each homomorphism of A onto the complex field admits a unique representing measure in the sense of* [6].

PROOF. Consider $X \subset \mathfrak{M}$. The Šilov boundaries of $\hat{A}$ and $\operatorname{Re} \hat{A}$ coincide (see [3]). Suppose that (a) and (b) hold, and notice that representing measures of a given $m \in \mathfrak{M}$ are identical with $\operatorname{Re} A$-harmonic measures in the sense of [3] belonging to the point m. It then suffices to apply Theorem 9 of [3].

Conversely, it is immediate that the Šilov boundary of $\hat{A}$ must be contained in X. On the other hand, given $x \in X$, Theorem 3 and (2.8) of [3] show that there is at least one representing measure for the homomorphism determined by the point x, which is carried by the Šilov boundary of $\hat{A}$, $\partial_{\hat{A}}$. By the uniqueness property, the latter measure must coincide with the Dirac measure at x. Hence $x \in \partial_{\hat{A}}$, and $\partial_{\hat{A}} = X$.

Next, we wish to show that all the results established in [6] for logmodular algebras hold for a sup-norm algebra satisfying condition (i) above. It will suffice to take up only those results in which logmodularity is actually used.

II. **The H^p theory.** We start with H^1 (see [6]), and the generalized Szegö-Kolmogoroff-Kreĭn theorem.

THEOREM 2. *Suppose that A satisfies condition* (i) *above. Let μ be the representing measure for $m \in \mathfrak{M}$. Let h be a non-negative function in $L^1(\mu)$. Then*

$$\inf_{f\in A_m} \int |1-f|^2 h d\mu = \exp\left[\int \log h d\mu\right]$$

where $A_m = \{f\in A: \int f d\mu = 0\}$.

PROOF. Lemma 4.5 of [6] is valid in the present context, so that we have

$$\exp\left[\int \log h d\mu\right] = \inf\left\{\int e^u h d\mu : u \in C_R(X),\ \int u d\mu = 0\right\},$$

$C_R(X)$ denoting as usual the algebra of all real-valued continuous functions on X. Consider $X \subset \mathfrak{M}$, and apply Theorem 1. The Dirichlet problem for a given $u \in C_R(X)$ admits a solution $\tilde{u} \in C_R(\mathfrak{M})$, and looking at Theorem 8 of [3] and its proof, we have

$$\int u d\mu = \tilde{u}(m) = \sup\left\{ \operatorname{Re}\hat{f}(m) = \int \operatorname{Re} f d\mu : f \in A,\ \operatorname{Re} f \leqq u \text{ on } X\right\}.$$

Thus if $\int u d\mu = 0$, we can find a sequence of $f_n \in A$, such that $\int \operatorname{Re} f_n d\mu \to 0$, and $\operatorname{Re} f_n \leqq u$ on X. By adding appropriate imaginary constants we may suppose that $\int \operatorname{Im} f_n d\mu = 0$; hence there is a sequence of real numbers d_n, such that $f_n + d_n \in A_m$, $d_n \to 0$, and $\operatorname{Re} f_n \leqq u$ on X. If we set $g_n = \exp 1/2(f_n + d_n)$, then $g_n \in A^{-1}$, where A^{-1} denotes the set of all invertible elements of A, and furthermore,

$$|g_n|^2 = \exp(\operatorname{Re} f_n + d_n) \leqq e^{d_n} e^u,$$

$$\int g_n d\mu = \hat{g}_n(m) = (\exp \tfrac{1}{2}(f_n + d_n))\hat{\ }(m) = \exp(\tfrac{1}{2}(f_n + d_n)\hat{\ }(m)),\ e^0 = 1,$$

$$\int |g_n|^2 h d\mu \leqq e^{d_n} \int e^u h d\mu.$$

Since $e^{d_n} \to 1$, we have

$$\begin{aligned} \inf&\left\{\int |g|^2 h d\mu : g \in A^{-1},\ \int g d\mu = 1\right\} \\ &\leqq \inf\left\{\int e^u h d\mu : u \in C_R(X),\ \int u d\mu = 0\right\} \\ &= \exp\left[\int \log h d\mu\right]. \end{aligned}$$

If $g \in A^{-1}$, $\int g d\mu = 1$, then $g = 1 - f$, $f \in A_m$. Hence we see that

$$\exp\left[\int \log h d\mu\right] \geqq \inf_{g\in A_m} \int |1 - g|^2 h d\mu.$$

From here on the proof may be finished as in [6], after noticing that Jensen's inequality holds under our hypothesis (it was already proved in this generality in [1]): If $g\in A_m$, $\int\log|1-g|^2 d\mu \geqq 2\log|\int(1-g)d\mu| = 0$; and by the convexity of the exponential function

$$\int |1 - g|^2 h d\mu \geqq \exp\left[\int (\log|1 - g|^2 + \log h) d\mu\right]$$
$$\geqq \exp\left[\int \log h d\mu\right].$$

This completes the proof.

Assuming condition (i), the proof of Theorem 4.3 of [6] remains valid without change. From this, and Theorem 2 above, we derive the following

THEOREM 3. *Assume A satisfies* (i), *$m\in\mathfrak{M}$, μ is the representing measure for m. Let $d\nu = hd\mu + d\nu_s$ be the Lebesgue decomposition of the positive measure ν on X, relative to μ. Then*

$$\inf_{g\in A_m} \int |1 - g|^2 d\nu = \exp\left[\int \log h d\mu\right].$$

This puts us in possession of the generalized Szegö theorem. Now as soon as we establish the following crucial fact, it will be possible to recover in our context the whole H^p theory of logmodular algebras, proceeding for the rest as in Hoffman [6].

THEOREM 4. *Let $g\in L^1(\mu)$, where μ is the representing measure of some $m\in\mathfrak{M}$. Suppose that $\int fg d\mu = 0$ for all $f\in A$, where A satisfies* (i). *Then*

$$\int \log|1 - g| d\mu \geqq 0.$$

SKETCH OF THE PROOF. We start as in the corresponding theorem for logmodular algebras, and for any $f\in A^{-1}$ obtain the inequality

$$\exp\left[\int \log |f| d\mu\right] \leqq \int |f|\, |1 - g| d\mu.$$

Now, for any given $u\in C_R(X)$, $\int u d\mu = 0$, proceeding as in the proof of Theorem 2, we find f_n and d_n with the same properties as in the latter proof, and set $g_n = \exp(f_n + d_n)$. Then $g_n\in A^{-1}$, and applying the inequality above,

$$\exp\left[\int \log |g_n| d\mu\right] = \exp\left[\int (\text{Re} f_n + d_n) d\mu\right] = 1$$

$$\leqq \int |\exp(f_n + d_n)| |1 - g| d\mu = e^{d_n} \int e^{\text{Re} f_n} |1 - g| d\mu$$

$$\leqq e^{d_n} \int e^u |1 - g| d\mu,$$

where $e^{d_n} \to 1$. Use of Lemmas 4.4 and 4.5 of [6] completes the proof. Using the theorem just proved and proceeding for the rest as in [6], we derive the following

THEOREM 5. *Let μ be as in Theorem* 4, *and suppose that A satisfies* (i). *Then $A + \bar{A}$ is dense in $L^2(\mu)$. $L^2(\mu) = H^2(\mu) \oplus \bar{H}^2_m(\mu)$, where $H^2(\mu)$ is the closure in $L^2(\mu)$ of A, and similarly H^2_m that of A_m. In particular, we have that* (i)$\Rightarrow$(ii).

III. **Maximal ideal space.** All the results of [6] concerning analytic structures in $\mathfrak{M}$ can be recovered under the present hypothesis. Below, we limit ourselves to prove mutual absolute continuity of the representing measures corresponding to two points of $\mathfrak{M}$ belonging to the same part, in the sense of Gleason:

THEOREM 6. *Suppose A satisfies* (i). *Let m, $m' \in \mathfrak{M}$ be such that $\|m - m'\| < 2$, and let μ, μ' be the corresponding representing measures. Then μ' is absolutely continuous with respect to μ, and $d\mu'/d\mu$ is bounded.*

SKETCH OF THE PROOF. Let K be an arbitrary positive constant. Suppose we do not have $\mu' \leqq K\mu$. Then there is a nonnegative $u \in C_R(X)$ such that

$$\int u d\mu' > K \int u d\mu.$$

Let $\tilde{u} \in C_R(\mathfrak{M})$ be the solution of the Dirichlet problem for u, so that $\tilde{u}(m) = \int u d\mu$, $\tilde{u}(m') = \int u d\mu'$, and from [3], we know that $\tilde{u}$ is the lower envelope of the functions $h = \text{Re}\, \hat{f}$, $f \in A$, such that $\text{Re}\, f \geqq u$ on X. Naturally $\tilde{u}(m) \geqq 0$, and we assume >0; the case $=0$ being treated similarly. Thus for $\epsilon > 0$, there is an h as above, such that $h(m') \geqq \tilde{u}(m')$, and $\tilde{u}(m) + \epsilon \geqq h(m) \geqq \tilde{u}(m) > 0$. Choose $K_1 > K$, so that $\tilde{u}(m') > K_1 \tilde{u}(m)$ still holds, and we obtain finally

$$h(m')/h(m) \geqq \tilde{u}(m')/(\tilde{u}(m) + \epsilon) > K_1/(1 + \epsilon(\tilde{u}(m))^{-1}).$$

Thus starting with an ϵ for which $K_1/(1 + \epsilon(\tilde{u}(m))^{-1}) > K$, we obtain an h satisfying $h(m') > Kh(m)$. Set $g = e^{-f}$, where $\operatorname{Re} \hat{f} = h$, h being the one we just determined. Then $g \in A^{-1}$, $\log |\hat{g}| = -\operatorname{Re} \hat{f}$, and we have

$$|m'(g)| < |m(g)|^K.$$

From here on, the proof can be completed as in Hoffman's paper.

IV. **Comments.** As one might suspect, the Szegö-Kolmogoroff-Kreĭn theorem is not only a consequence of the assumption that A satisfies (i), but in a sense characteristic of such algebras. Moreover, "locally," i.e. for any one $m \in \mathfrak{M}$, the validity of the Szegö theorem is equivalent to the uniqueness of the representing measure for m, in the sense described below:

THEOREM 7. *Let $m \in \mathfrak{M}$. Then the generalized Szegö-Kolmogoroff-Kreĭn theorem holds with respect to any representing measure μ of m, if and only if m admits a unique representing measure.*

SKETCH OF THE PROOF. It remains to prove the "only if" part. Let μ, ν be two representing measures for m; and let $d\nu = hd\mu + d\nu_s$ be the Lebesgue decomposition of ν with respect to μ. Then $\eta = (1-t)\nu + t\mu$ is again a representing measure of m, for $0 \leqq t \leqq 1$, with absolutely continuous part $((1-h)+t)d\mu$. It is easily seen that for any representing measure η of m, one has $\inf\{\int |1-f|^2 d\eta : f \in A_m\} = 1$. Hence, because of our assumption that the Szegö theorem holds, we have identically

$$\int \log ((1-t)h + t)d\mu = 0 \qquad \text{for } 0 \leqq t \leqq 1.$$

Using now the Taylor expansion (with second order remainder) for $F(t) = \log((1-t)h+t)$, where only the variable (parameter) t is explicitly shown, between t and $t+s$, t and $t-s$ (s small for fixed t), and finally integrating, we obtain

$$\int (F''(t+s_1) + F''(t-s_2))d\mu = 0, \qquad 0 \leqq s_1, s_2 \leqq |s|,$$

where $F''(t) = (1-h)^2/((1-t)h+t)^2$. One can check that $F''(t+s)$ is essentially bounded, uniformly for small s, and fixed $t > 0$ and < 1. Thus application of the Lebesgue dominated convergence theorem, for $s \to 0$, yields finally

$$\int (1-h)^2/((1-t)h+t)^2 d\mu = 0 \qquad \text{for } 0 < t < 1.$$

Hence $h = 1$ almost everywhere with respect to μ, and $d\nu = d\mu + d\nu_s$.

Interchanging the roles of μ and ν, we have $d\mu = d\nu + d\mu_s$. It follows that

$$d\mu_s + d\nu_s = 0,$$
$$d\mu = d\nu.$$

In a recent letter, K. Hoffman kindly calls to our attention several interesting facts:

(1) Although the earlier mentioned ideas, behind the above development, are suggested by an abstract Dirichlet problem, one can easily work without any explicit mention of the latter. In fact, technically all one uses from it is the condition under which a Hahn-Banach extension is unique, applied to the case of a positive functional on a subspace of $C_R(X)$; this, under the form of an appropriate lemma, can be used where needed, in the above proofs, instead of the solution of some Dirichlet problem.

(2) Most of what we prove is of a "local" character, in the sense that only the assumption of (i) for any single $m \in \mathfrak{M}$, instead of all, is involved, and one really obtains theorems about $H^p(m)$, making the general H^p theory independent of any global condition on A. Moreover, Hoffman indicates in his letter how a rather complete theory for $H^p(m)$ can be obtained if one combines our above approach with recent unpublished results of several people.

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