ON THE CONTINUITY OF INVERSE OPERATORS IN (LF)-SPACES[1,2]

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Though the well-known equivalence of continuity of the inverse operator and permanent solvability of the adjoint one can be verified for a very wide class of linear topological spaces, it often turns out to be not a very deep fact and one can hardly expect to get much help using the equivalence to solve problems out of the classical analysis.

This paper gives another characterization of the permanent solvability of the adjoint operator, which seems to be less trivial and involving more of the special internal structure of (LF)-spaces. Proofs of the results presented here will appear in the Studia Mathematicae.

Consider an (LF)-space (X, τ) [1]. A subspace of X which is metrizable and complete provided with the relativization of topology τ is called an (F)-*subspace* of (X, τ); (X, τ) is called an (LB)-*space* iff every (F)-subspace of (X, τ) is a Banach space. We say that an (LF)-space (Z, ρ) *majorizes* an (LF)-space (X, τ) iff Z is algebraically a subspace of X and to every (F)-subspace U of (Z, ρ) there corresponds an (F)-subspace Y of (X, τ) such that $U \subset Y$ and the natural injection of (U, ρ) into (Y, τ) is continuous (we denote the relativizations of topologies by the same letters).

A *projective component* (ξ) of an (LF)-space (X, τ) is an (LB)-space (X_ξ, ξ) such that (X, τ) majorizes (ξ) and every (F)-subspace of (ξ) is contained in a ξ-closure of an (F)-subspace of (X, τ); the projective component (ξ) is *reflexive* iff every (F)-subspace of (ξ) is a reflexive Banach space. A family Φ of projective components of (X, τ) form a *basis* of projective components of (X, τ) iff to every τ-continuous pseudonorm $|\cdot|$ on X there correspond $(\xi) \in \Phi$ such that for every (F)-subspace U of (X, τ) the identical injection of (U, τ) into $(U, |\cdot|)$ is continuous and to every (ξ_1), $(\xi_2) \in \Phi$ there correspond $(\xi_3) \in \Phi$ such that (ξ_3) majorizes both (ξ_1) and (ξ_2).

Let (X, τ) and (X_1, τ_1) be two (LF)-spaces with bases of projective components Φ and Φ_1 respectively and let A be a continuous mapping of (X_1, ξ_1) into (X, ξ).

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PROPOSITION 1. *If the adjoint mapping* A' *maps* $(X_1, \tau_1)'$ *onto* $(X, \tau)'$, *then* A *is one-to-one and to every* $(\xi_1)\in\Phi_1$ *there correspond* $(\xi)\in\Phi$ *such that* $B=A^{-1}$ *defined on* $X_0=AX_1$ *satisfies the following condition*:

(*) *To every* (F)-*subspace* Y *of* (X, τ) *there correspond an* (F)-*subspace* S *of* (ξ_1) *such that* $B(X_0\cap Y)\subset S$ *and* B *is continuous from* $(X_0\cap Y, \xi)$ *to* (S, ξ_1).

The proof of Proposition 1 is based on the results of [2] and [3]. Let (ξ) be a projective component of (X, τ) and let U be an (F)-subspace of (ξ) and Y an (F)-subspace of (X, τ). Since $U\oplus Y$ can be identified with $U\times Y/\{(x, -x): x\in U\cap Y\}$, we can introduce in $U\oplus Y$ the quotient topology $\xi\wedge\tau$ of the usual product topology of the topologies ξ and τ relativized to U and Y respectively. Consider an (LF)-space (X, τ) and a projective component (ξ) of (X, τ). Assume that (X, τ) admits a basis of reflexive projective components. Take a subspace X_0 of X that has τ-closed intersections with all (F)-subspaces of (X, τ).

PROPOSITION 2. *In order that every functional* z_0' *defined on* X_0 *and* ξ-*continuous on every* (F)-*subspace of* (X, τ) *contained in* X_0 *admits an extension to some* $z'\in(X, \tau)'$ *it is necessary and sufficient that there exists a projective component* (ζ) *of* (X, τ), *majorizing* (ξ), *such that the following condition is fulfilled*:

(**) *To every* (F)-*subspace* U *of* (ζ) *there corresponds an* (F)-*subspace* V *of* (ξ), $V\supset U\cap X$, *such that for every* (F)-*subspace* Y *of* (X, τ) *we have*

$$U\cap \mathrm{Cl}_{\tau\wedge\zeta}(X_0\cap Y)\subset \mathrm{Cl}_{\xi}(X_0\cap V),$$

where $\mathrm{Cl}_{\tau\wedge\zeta}$ *denotes the closure in* $(Y\oplus U, \tau\wedge\zeta)$ *and* Cl_{ξ} *denotes the closure in* (ξ).

The proof is based on the results of [2] and [3]. Consider a continuous mapping A of an (LF)-space (X_1, τ_1) into an (LF)-space (X, τ) admitting a basis of reflexive projective components. Let in the following Φ and Φ_1 be bases of projective components in (X, τ) and (X_1, τ_1) respectively.

THEOREM. *In order that the adjoint mapping* A' *maps* $(X, \tau)'$ *onto* $(X_1, \tau_1)'$ *it is necessary and sufficient that the following conditions are satisfied.*

(i) *The mapping* A *is one-to-one and to every* $(\xi_1)\in\Phi_1$ *there correspond* $(\xi)\in\Phi$ *such that* A^{-1} *satisfies* (*) *with* (ξ) *and* (ξ_1).

(ii) *To every* $(\xi)\in\Phi$ *there correspond* $(\zeta)\in\Phi$ *majorizing* (ξ) *such that* (**) *holds for* (ξ), (ζ) *and* $X_0=AX_1$.

The theorem is an easy consequence of Propositions 1 and 2. Since in majority of cases the continuity of the inverse operator is necessary and sufficient for A' to be a mapping "onto," the conditions (i) and (ii) usually describe the continuity of A^{-1} in the relative topology of the subspace X_0 of (X, τ). This takes place, for instance, in the space $\mathfrak{D}$ of infinitely differentiable functions with compact supports, defined over a fixed open subset of the n-dimensional Euclidean space. The space $\mathfrak{D}$ is considered provided with the usual topology which we denote by ρ.

For $(\mathfrak{D}, \rho)$ one can easily verify the existence of bases of reflexive projective components. Conveniently, several important bases of projective components of $(\mathfrak{D}, \rho)$ can be written explicitly.

References

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