ON THE STRUCTURE OF SEMI-NORMAL OPERATORS[1]

BY C. R. PUTNAM

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1. **Preliminaries.** Only bounded operators on a Hilbert space $\mathfrak{H}$ of elements x will be considered. If A is self-adjoint with the spectral resolution

(1) $A = \int \lambda dE(\lambda)$,

and if $\mathfrak{H}_a = \mathfrak{H}_a(A)$ denotes the set of elements x for which $\|E(\lambda)x\|^2$ is an absolutely continuous function of λ, then $\mathfrak{H}_a$ is a subspace; cf. [2, p. 240], [3, p. 436] and [6, p. 104]. If $\mathfrak{H} = \mathfrak{H}_a$, then A is called absolutely continuous. The one-dimensional Lebesgue measure of the spectrum of a self-adjoint operator A will be denoted by meas sp(A).

An operator T on $\mathfrak{H}$ is called semi-normal if

(2) $TT^* - T^*T \equiv D \geqq 0$ or $D \leqq 0$.

There will be proved the following result concerning such an operator.

2. **Theorem.** *If T satisfies* (2) *and if $\mathfrak{M} = \mathfrak{M}_T$ is the smallest subspace of $\mathfrak{H}$ reducing T and containing the range of D, then*

(3) *$T + T^*$ is absolutely continuous on $\mathfrak{M}$,*

and, if $\mathfrak{M}^\perp$ denotes the orthogonal complement of $\mathfrak{M}$ (so that $\mathfrak{M}^\perp$ also reduces T), then

(4) *T is normal on $\mathfrak{M}^\perp$.*

In addition,

(5) $2\pi\|D\| \leqq \|T - T^*\|$ meas sp$(T + T^*)$,

and the inequality (5) *is optimal in the sense that there exist examples with $D \neq 0$ for which* (5) *becomes an equality.*

As a consequence, if T is semi-normal but not normal, then $\mathfrak{H}_a(T + T^*) \neq 0$, a result which can also be concluded from [4, Corollary 3, p. 1029], where the symbol "<" should be replaced by "$\neq$." (This same Corollary, incidentally, also implies the result proved by Andô [1] that a completely continuous semi-normal operator T must be normal. In fact, if T is completely continous, so also are T^* and $T + T^*$. But the spectrum of $T + T^*$ clearly must be of measure zero.)

If θ is real and $T(\theta) = e^{i\theta}T$, then (2) is unchanged if T is replaced by $T(\theta)$. Also, it is clear that the set $\mathfrak{M}_{T(\theta)}$ is independent of θ. It follows that (3), (4) and (5) remain valid if, in each instance, T is

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replaced by $T(\theta)$. In particular then, relations (3) and (5) become assertions concerning the absolute continuity and spectra of both the real and the imaginary parts of a semi-normal operator T.

The proof of the Theorem will depend upon results proved in [5] and which will be stated here, in a form convenient for application, as a

LEMMA. *Let H and J be self-adjoint operators and suppose that*

(6) $HJ-JH=iC$, *where* $C\geqq 0$ *or* $C\leqq 0$.

Then,

(7) $\mathfrak{L}\subset\mathfrak{H}_a(H)$,

where $\mathfrak{L}$ denotes the smallest subspace reducing both H and J and also containing the range of C. Furthermore,

(8) $\pi\|C\|\leqq\|J\|$ meas sp(H).

It is clear from the symmetry of the condition (6) that (7) and (8) remain true if H and J are interchanged.

3. **Proof of the Theorem.** Let T be represented as

(9) $T=H+iJ$, where $H=(T+T^*)/2$ and $J=(T-T^*)/2i$,

so that (2) and (6) are equivalent by virtue of (9) and

(10) $D=2C$.

It is clear that the space $\mathfrak{L}$ of the Lemma must then coincide with the space $\mathfrak{M}$ of the Theorem. Relations (3) and (5) now follow respectively from (7) and (8), while relation (4) is a consequence of the fact that $\mathfrak{M}^\perp$ is contained in the null space of D. An example involving finite interval Hilbert transforms was given in [5] for which the hypothesis of the Lemma is fulfilled and for which (8) becomes an equality (with $C\neq 0$). This result in turn yields, by virtue of (9) and (10), an example in which equality holds in (5) and $D\neq 0$.

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PURDUE UNIVERSITY