THE COHOMOLOGY OF GROUP EXTENSIONS

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1. **Introduction.** Suppose Π is a group with a finitely generated abelian normal subgroup M and let $\Phi=\Pi/M$, i.e. Π satisfies the exact sequence

$$(*)\qquad 0\to M\to\Pi\to\Phi\to 1.$$

The isomorphism class of Π is determined by (A) the groups M and Φ, (B) the structure of M as a Φ-module, and (C) a cohomology class $a\in H^2(\Phi;\ M)$ which describes the extension (cf. [1]). In principle then it should be possible to compute $H^*(\Pi)$, the cohomology ring of Π, from the above information. Practically, however, this seems to be impossible in general even if we assume known the cohomology of M and Φ. Our objective here is to solve an approximation to this problem.

The Hochschild-Serre spectral sequence [2] provides us with a sequence of differential rings $(E_r,\ d_r)$ $(r=1,\ 2,\ \cdots)$ which approximate the ring $H^*(\Pi)$ and such that $E_{r+1}=H(E_r,\ d_r)$. Hochschild and Serre computed E_2 and found that $E_2^{p,q}\cong H^p(\Phi;\ H^q(M))$. So E_2 depends only on (A) and (B) and is therefore a rather crude approximation to $H^*(\Pi)$. We determine d_2 (and hence E_3) in terms of (A), (B), and (C). Hochschild and Serre found d_2 on $E_2^{*,1}$ ("the first row"), and our results can be thought of as a generalization of theirs. We assume we have coefficients in a field F although the results are valid in somewhat greater generality.

In §2 we generalize a technique in [2] and define two new spectral sequences $\hat{E}_r$ and $\overline{E}_r$ and a cup product pairing from $\hat{E}_r\otimes\overline{E}_r$ to E_r. The problem of computing d_2 in E_2 is reduced to computing $\hat{d}_2$ on a sequence of classes $f^n\in\hat{E}_2^{n,0}$, and then the value of d_2 on a class in $E_2^{n,p}$ is equal to the cup product of $b^n=\hat{d}_2(f^n)$ and an appropriate class in $\overline{E}_2^{0,p}$.

In §3 we assume that $(*)$ splits or equivalently that $a=0$. In this case the entire spectral sequence $(\mathfrak{E}_r,\ \mathfrak{d}_r)$ depends only on (A) and (B). The classes $v^n=\mathfrak{d}_2(f^n)$ obtained in this case are called characteristic classes of the Φ-module M. They provide some measure of the difference between the cohomology of the split extension $\Phi\cdot M$ and that of the direct product $\Phi\times M$.

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§4 shows that the general case can be obtained from the special case by adding a correction term to v^n, i.e. $b^n = v^n + a^n$. The correction term a^n is determined by a and Pontryagin multiplication in $H_*(M)$.

Proofs and applications will appear in subsequent papers. We have used these results (especially Remark 2 in §3) to compute the cohomology of certain flat Riemannian manifolds.

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2. We will omit writing the coefficient group when it is the field F. Since M is a Φ-module, $H_n(M)$ and $H^n(M)$ are also Φ-modules and if we consider M to act on them trivially, they become Π modules in a natural way. We fix n, and let (E_r, d_r) (respectively $(\hat{E}_r, \hat{d}_r)$; respectively $(\bar{E}_r, \bar{d}_r)$) be the spectral sequence for (*) with coefficients F (respectively $H_n(M)$; respectively $H^n(M)$). There is a canonical isomorphism $\theta: E_2^{p,n} \to \bar{E}_2^{p,0}$ since

$$H^p(\Phi; H^n(M)) \cong H^p(\Phi; H^0(M; H^n(M))).$$

Since $H^n(M) \cong \mathrm{Hom}(H_n(M), F)$, evaluation gives a pairing from $H_n(M) \otimes H^n(M)$ to F which induces a cup product pairing from $\hat{E}_r^{p,q} \otimes \bar{E}_r^{s,t}$ to $E_r^{p+s,q+t}$. Now

$$\hat{E}_2^{0,n} \cong H^0(\Phi; H^n(M; H_n(M))) \cong \mathrm{Hom}_\Phi(H_n(M), H_n(M)).$$

Let $f^n \in \hat{E}_2^{0,n}$ correspond to the identity map.

LEMMA A. *Let* $u^n \in E_2^{p,n}$. *Then*

$$u^n = f^n \cup \theta(u^n).$$

3. Let $\Phi \cdot M$ be the split extension, i.e. $\Phi \cdot M$ satisfies the split exact sequence

$$(**) \qquad 0 \to M \to \Phi \cdot M \leftrightarrows \Phi \to 1.$$

Let $(\mathfrak{E}_r, \mathfrak{d}_r)$ be the spectral sequence for (**) with coefficients $H_n(M)$. Since the second term of the spectral sequence is independent of the extension, $\hat{E}_2 \cong \mathfrak{E}_2$, and we can consider $f^n \in \mathfrak{E}_2^{0,n}$.

DEFINITION. Let $v^n = \mathfrak{d}_2(f^n) \in \mathfrak{E}_2^{2,n-1} = \hat{E}_2^{2,n-1}$. We call v^n the nth *characteristic class* of the Φ-module M.

REMARKS. (1) v^1 is always 0.

(2) If Φ is a cyclic group of prime order, and M is torsionfree as an abelian group, then $v^n = 0$ for all n. The proof of this apparently difficult fact uses the $Z[M]$-free resolution of Z described in [4] and the knowledge of the indecomposable Φ-modules [3].

(3) If Φ is Z_2 and M is Z_8 and the generator of Φ takes a generator of M into five times itself, then $v^2 \neq 0$.

(4) Since v^n depends only on the Φ-module M, it is not surprising that v^n can be defined without reference to a spectral sequence.

4. Returning to the general case (*), recall $a \in H^2(\Phi;\ M) \cong H^2(\Phi;\ H_1(M;\ Z))$. Let $\chi\colon Z \to F$ send 1 into 1. χ induces χ_*: $H^2(\Phi;\ H_1(M;\ Z)) \to H^2(\Phi;\ H_1(M))$. Let $a' = \chi_*(a)$. Now Pontryagin multiplication gives a homomorphism $H_1(M) \otimes H_{n-1}(M) \to H_n(M)$, or, equivalently a homomorphism $H_1(M) \to \mathrm{Hom}(H_{n-1}(M),\ H_n(M)) \cong H^{n-1}(M;\ H_n(M))$. We define $a^n \in H^2(\Phi;\ H^{n-1}(M;\ H_n(M))) = \hat{E}_2^{2,n-1}$ to be the image of $-a'$ under this coefficient homomorphism.

LEMMA B. $\hat{d}_2(f^n) = \mathfrak{d}_2(f^n) + a^n$, *i.e.*

$$b^n = v^n + a^n.$$

THEOREM. $d_2(u^n) = b^n \cup \theta(u^n) = (a^n + v^n) \cup \theta(u^n)$.

PROOF. Using Lemma A we have

$$d_2(u^n) = \bar{d}_2(f^n) \cup \theta(u^n) + (-1)^n f^n \cup \hat{d}_2(\theta(u^n)).$$

Since $\theta(u^n) \in \bar{E}_2^{p,0}$, Lemma B completes the proof.

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