3. B. R. Gelbaum, *Tensor products of Banach algebras*, Canad. J. Math. 11 (1959), 297–310.
4. J. Tomiyama, *Tensor products of commutative Banach algebras*, Tôhoku Math. J. 12 (1960), 147–154.
5. R. Schatten, *A theory of cross spaces*, Annals of Mathematics Studies No. 26, Princeton Univ. Press, Princeton, N. J., 1950.
6. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16, 1955.
7. J. Gil de Lamadrid, *Measures and tensors*, (to appear).
8. L. Loomis, *Introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.

UNIVERSITY OF MINNESOTA

AN INTEGRATION-BY-PARTS FORMULA[1]

BY J. S. MAC NERNEY

Communicated by Walter Rudin, August 5, 1963

In 1914, W. H. Young [4] introduced a modification of the Riemann-Stieltjes integral which, for functions F and G defined on the real line with G of bounded variation on each interval and F suitably restricted, yields an additive interval function:

$$(Y)\int_a^b F\cdot dG + (Y)\int_b^c F\cdot dG = (Y)\int_a^c F\cdot dG.$$

In 1959, T. H. Hildebrandt [1] published a study of a certain linear initial-value problem involving these Young integrals, which extended some of the earlier results of H. S. Wall and of the present author (see [2] for discussion and references). In 1962, there was discovered a connection between the Young integral and the interior integral as introduced by S. Pollard in 1920 [3], *viz.*, the systems

$$U(x) = U(c) + (Y)\int_c^x U\cdot dH \quad \text{and} \quad V(x) = V(c) + (I)\int_x^c dH\cdot V,$$

with H a function from the real line to a complete normed ring, are naturally adjoint to one another [2, p. 326]. Both integrals are to be interpreted as limits in the sense of successive refinements of subdivisions of the interval of integration.

Suppose each of F and G is a function from the real line to the complete normed ring N. If each of F and G is of bounded variation

---

[1] Presented to the Society, July 18, 1963.

on the interval $[a, b]$ then each of $(Y)\int_a^b F\cdot dG$ and $(I)\int_a^b dF\cdot G$ is known to exist. Hence, the latter integral exists under the condition that F is of bounded variation on $[a, b]$ and G is quasicontinuous, *i.e.*, each of the limits $G(x-)$ and $G(x+)$ exists for each number x. Here is a new connection between these integrals, which also provides a new existence theorem for the former one.

THEOREM A. *If each of F and G is a function from the real line to the complete normed ring N, F is of bounded variation on the interval $[a, b]$, and G is quasicontinuous, then*

$$(Y)\int_a^b F\cdot dG + (I)\int_a^b dF\cdot G = F(b)G(b) - F(a)G(a).$$

INDICATION OF PROOF. If $a \leqq x < y < z \leqq b$ then

$$\begin{aligned} W = {} & F(x)[G(x+)-G(x)]+F(y)[G(z-)-G(x+)]+F(z)[G(z)-G(z-)] \\ & +[F(z)-F(x)]G(y)-[F(z)G(z)-F(x)G(x)] \\ = {} & [F(x)-F(z)][G(x+)-G(y)]+[F(z)-F(y)][G(x+)-G(z-)], \end{aligned}$$

so that one has the estimate

$$|W| \leqq 2\left(\int_x^z |dF|\right)(\text{L.U.B. } |G(v)-G(u)| \text{ for } x<u<v<z).$$

ADDENDUM. As has been observed by Randolph Constantine (an oral communication in seminar), the hypotheses on F and G in Theorem A can be interchanged. To see this, one first notes the identity

$$\begin{aligned} & [F(z)-F(x)]G(y) \\ & \quad = F(z)G(z)-F(x)G(x)-F(x)[G(y)-G(x)]-F(z)[G(z)-G(y)]; \end{aligned}$$

next, if H is a simple step-function and $\{t_p\}_0^{2n}$ is an increasing numerical sequence with $t_0=a$ and $t_{2n}=b$,

$$\begin{aligned} & \left|\sum_1^n [F(t_{2p})-F(t_{2p-2})]G(t_{2p-1}) - \sum_1^n [H(t_{2p})-H(t_{2p-2})]G(t_{2p-1})\right| \\ & \qquad \leqq |F-H|_{[a,b]}\left(|G(a)| + |G(b)| + \int_a^b |dG|\right), \end{aligned}$$

where $|F-H|_{[a,b]} = \text{L.U.B. } |F(u)-H(u)|$ for u in $[a, b]$, and also

$$\left|(Y)\int_a^b F\cdot dG - (Y)\int_a^b H\cdot dG\right| \leqq |F-H|_{[a,b]}\left(\int_a^b |dG|\right).$$

Thus, an argument is easily made to establish the following somewhat stronger theorem.

THEOREM B. *If each of F and G is a quasicontinuous function from the real line to the complete normed ring N, and one of F and G is of bounded variation on the interval $[a, b]$, then*

$$(Y)\int_a^b F\cdot dG + (I)\int_a^b dF\cdot G = F(b)G(b) - F(a)G(a).$$

REMARK. The reader is invited to contrast this formula with the corresponding result involving Young integrals alone (or interior integrals alone), as obtained by Hildebrandt [1, p. 355] for the case that both F and G are of bounded variation. For this case, there is a more general result available, as indicated in the following theorem.

THEOREM C. *If Axioms* I *and* II [2, p. 321] *hold, each of F and G is a function from the interval $[a, b]$ to N, and $dG(x, z) = K_1[1](x, z)$ and $dF(x, z) = K_2[1](x, z)$ for $a \leqq x < z \leqq b$, then*

$$\begin{aligned} K_1[F](a, b) + K_2[G](a, b) &= F(b)G(b) + \sum_{a<z\leqq b}\{dF\cdot K_1[1_z] + K_2[1_z]\cdot dG - dF\cdot dG\}(z-, z) \\ &\quad - F(a)G(a) - \sum_{a\leqq x<b}\{dF\cdot K_1[1_x] + K_2[1_x]\cdot dG - dF\cdot dG\}(x, x+). \end{aligned}$$

REMARK. After obtaining the preceding results, the author learns (July 27, 1963) that Theorem B has been discovered by T. H. Hildebrandt (on May 28, 1963) for *numerical valued functions F* and G: that priority of discovery is hereby cordially acknowledged to Professor Hildebrandt.

BIBLIOGRAPHY

1. T. H. Hildebrandt, *On systems of linear differentio-Stieltjes-integral equations*, Illinois J. Math. **3** (1959), 352–373.

2. J. S. Mac Nerney, *A linear initial-value problem*, Bull. Amer. Math. Soc. **69** (1963), 314–329.

3. S. Pollard, *The Stieltjes integral and its generalizations*, Quart. J. Pure and Appl. Math. **49** (1920), 73–138.

4. W. H. Young, *On integration with respect to a function of bounded variation*, Proc. London Math. Soc. (2) **13** (1914), 109–150.

THE UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL