HILBERT ALGEBRAS WITH IDENTITY

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With a Hilbert algebra with identity (HAI) we mean a Hilbert space with inner product (x, y) which is also an associative Banach algebra with identity e; the norm $\|x\| = (x, x)^{1/2}$ satisfying

$$\|xy\| \leqq \|x\| \cdot \|y\|, \tag{1}$$

$$\|e\| = 1. \tag{2}$$

An HAI is called *real* if it is a real Hilbert space and a real algebra; *complex* if it is complex in both respects.

As a consequence of a result on the geometric properties of the unit sphere in Banach algebras, originally due to Bohnenblust and Karlin [2], one easily gets

THEOREM 1. *A complex Hilbert algebra with identity is isomorphic to the complex numbers.*

(This is a rephrasing of [3, Corollary 2, p. 25].)

In connection with this, it was conjectured by I. Kaplansky[1] that every real HAI must be isomorphic to the reals, complexes or quaternions. The object of this note is to prove that this is true. (Of course if condition (2) or the assumption of identity is dropped there are many other examples.) In particular, we will see that the given conditions imply that the norm must satisfy

$$\|xy\| = \|x\| \cdot \|y\|,$$

in other words be an *absolute value.*

The proof depends partly on techniques developed in [3]. We start with two preliminary results.

PROPOSITION 1. *For an element x in a real HAI the conditions*

$$1^\circ \qquad (e, x) = 0,$$

$$2^\circ \qquad \|\exp \alpha x\| = 1 \quad \textit{for all real } \alpha$$

are equivalent.

PROOF. We define

[1] Personal letter, April, 1963. I want to thank Professor Kaplansky for directing my attention to this enjoyable problem.

$$\psi(x) = \max_{\theta=\pm 1} \lim_{\alpha\to+0} \alpha^{-1}(\|e + \alpha\theta x\| - 1) \tag{3}$$

and will show that 1° and 2° are both equivalent to $\psi(x)=0$. (This condition means, geometrically, that the line $e+\alpha x$ is a tangent of the unit sphere at e.) Straightforward computation gives $\psi(x) = |(e, x)|$, hence 1° is equivalent to $\psi(x)=0$.

But we also have (see [3, p. 25]) that

$$\psi(x) = \max_{\theta=\pm 1} \lim_{\alpha\to+0} \alpha^{-1} \log\|\exp \alpha\theta x\|. \tag{4}$$

Assume that $\psi(x)=0$ and let $h(\alpha)=\log\|\exp \alpha x\|$. Then h is a subadditive function on the real line with $h(0)=0$ and non-positive right and non-negative left derivative at 0. Such a function must be identically 0 [3, p. 24]. Thus $\psi(x)=0$ implies $\|\exp \alpha x\|=1$ and since the reverse implication is immediate from (4) the proof is complete.

It is now clear that we can express each element x uniquely as $x=\xi e+x'$ where $(e, x')=0$ and $\|\exp \alpha x'\|=1$ for all α.

PROPOSITION 2. *In a real HAI there are no topologically nilpotent elements except* 0.

PROOF. Assume that $\lim_{n\to\infty} \|x^n\|^{1/n}=0$ for some x. Since

$$\|\exp \alpha x\| \leqq \sum_{n=0}^{\infty} \frac{|\alpha|^n}{n!} \|x^n\| \tag{5}$$

we see, by comparing coefficients, that

$$\|\exp \alpha x\| = O(\exp \delta|\alpha|), \qquad |\alpha| \to \infty, \tag{6}$$

for every $\delta>0$. Assume now that $x=\xi e+x'$ with $(e, x')=0$. Then, from Proposition 1,

$$\|\exp \alpha x\| = \|\exp \alpha(\xi e + x')\| = \exp \alpha\xi\cdot\|\exp \alpha x'\| = \exp \alpha\xi. \tag{7}$$

From (6) and (7) it follows that $\xi=0$ and hence $\|\exp \alpha x\|=1$. If f is a continuous linear functional the function

$$\varphi\colon \varphi(\alpha) = f(\exp \alpha x) = f(e) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} f(x^n)$$

can be continued analytically to an entire function $\bar{\varphi}$ in the complex plane. Estimates analogous to (5) and (6) show that $\bar{\varphi}$, as an entire function, is at most of order 1, minimum type. Since it is bounded on the real axis a Phragmén-Lindelöf theorem [1, p. 84] tells that $\bar{\varphi}$ is a constant. Hence $f(x)=0$ for every f and $x=0$.

THEOREM 2. *A real Hilbert algebra with identity is isomorphic to the real numbers, the complex numbers or the quaternions.*

PROOF. We take an $x \neq 0$ and let C_x be the closed subalgebra spanned by x and e. From Proposition 2 follows that C_x is semi-simple, hence isomorphic to an algebra $\hat{C}_x$ of continuous functions on a compact space ϕ under a map $y \rightarrow \hat{y} = \hat{y}(\varphi)$. An element $y \in C_x$ has inverse in C_x if and only if $\hat{y}(\varphi) \neq 0$ for all $\varphi \in \phi$. If $y = \eta e + y'$ with $(e, y') = 0$ we have $\hat{y}(\varphi) = \eta + (y')^\frown(\varphi)$. Since $\exp \alpha y'$ is bounded, $\exp \alpha (y')^\frown$ is also bounded and $(y')^\frown$ has only imaginary values. Thus the functions in $\hat{C}_x$ have constant real parts. If $x = \xi e + x'$ we have

$$\hat{x}(\varphi) = \xi + (x')^\frown(\varphi)$$

and if $\xi \neq 0$, $\hat{x}(\varphi) \neq 0$ and x has an inverse. If $\xi = 0$ we must have $(x')^\frown(\varphi) \not\equiv 0$ (since $x \neq 0$), but also $(x^2)^\frown(\varphi) = \hat{x}^2(\varphi) = (x')^{\frown 2}(\varphi)$ $=$ real, since $(x')^\frown$ is imaginary-valued. Then $(x^2)^\frown$ is a nonzero constant, x^2 has inverse and x has inverse.

Thus we have shown that every $x \neq 0$ has an inverse. Since the only normed real division algebras are the reals, the complexes and the quaternions the theorem is proved.

Utilizing Proposition 1, it is a simple matter to verify that the familiar norms for the complexes and quaternions are unique as real HAI norms, and so a given HAI norm, satisfying (1) and (2), is in fact an absolute value.

REMARK. In Theorems 1 and 2 we need not assume the algebra to be complete. If A satisfies all the axioms for an HAI except completeness, its completion (as a normed space) is an HAI and hence, according to Theorem 1 or 2, finite-dimensional. Then A is also finite-dimensional and automatically complete.

REFERENCES

1. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954.

2. H. F. Bohnenblust and S. Karlin, *Geometrical properties of the unit sphere of Banach algebras*, Ann. of Math. (2) **62** (1955), 217–229.

3. L. Ingelstam, *A vertex property for Banach algebras with identity*, Math. Scand. **11** (1962), 22–32.

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