

# FUNDAMENTAL SOLUTIONS OF INVARIANT DIFFERENTIAL OPERATORS ON SYMMETRIC SPACES

BY S. HELGASON<sup>1</sup>

Communicated by I. M. Singer, June 27, 1963

**1. Introduction and notation.** Let  $S$  be a Riemannian globally symmetric space,  $G$  the largest connected group of isometries of  $S$  in the compact open topology. We assume that  $S$  is of the noncompact type, that is,  $G$  is semisimple and has no compact normal subgroup  $\neq \{e\}$ . Let  $o$  be any point in  $S$ ,  $K$  the isotropy subgroup of  $G$  at  $o$ ,  $\mathfrak{k}$  and  $\mathfrak{g}$  their respective Lie algebras, and  $\mathfrak{p}$  the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$  with respect to the Killing form  $B$  of  $\mathfrak{g}$ . Let  $\mathfrak{a}$  be any maximal abelian subspace of  $\mathfrak{p}$  and let  $A = \exp(\mathfrak{a})$ . For each  $\lambda$  in the dual space of  $\mathfrak{a}$  (which we identify with  $\mathfrak{a}$ , via  $B$ ) let  $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$ . Let  $d_\lambda = \dim(\mathfrak{g}_\lambda)$ . Choose some order on  $\mathfrak{a}$  and let

$$\mathfrak{n} = \sum_{\lambda > 0} \mathfrak{g}_\lambda, \quad \rho = \sum_{\lambda > 0} d_\lambda \lambda, \quad \pi' = \prod_{\lambda > 0} \lambda^{d_\lambda}$$

and let  $\pi$  denote the product of the distinct prime factors in  $\pi'$ . Then we have the Iwasawa decompositions  $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ ,  $G = KAN$  where  $N$  is the nilpotent group  $\exp(\mathfrak{n})$ . Given  $g \in G$ , let  $H(g)$  denote the unique element in  $\mathfrak{a}$  for which  $g \in K \exp H(g)N$ . Let  $W$  denote the Weyl group  $M'/M$  where  $M$  and  $M'$ , respectively, denote the centralizer and normalizer of  $\mathfrak{a}$  in  $K$ .

For each  $\lambda \in \mathfrak{a}$  consider the spherical function

$$\phi_\lambda(x) = \int_K e^{(i\lambda - \rho)(H(xk))} dk \quad (x \in G)$$

$dk$  being the normalized Haar measure on  $K$ . Let  $c(\lambda)$  denote Harish-Chandra's function on  $\mathfrak{a}$  which occurs in the leading term of the asymptotic expansion of  $\phi_\lambda$  [2, p. 283], i.e.,

$$\phi_\lambda(\exp H) \sim \sum_{s \in W} c(s\lambda) e^{(is\lambda - \rho)(H)}$$

where  $\lambda$  and  $H$  are suitably restricted in  $\mathfrak{a}$ .

Each  $x \in G$  can be written uniquely in the form  $x = k \exp X$  ( $k \in K$ ,  $X \in \mathfrak{p}$ ). We put  $|X| = (B(X, X))^{1/2}$  and  $\omega(X) = \{\det (\sinh ad X/ad X)_\mathfrak{p}\}^{1/2}$  where the subscript  $\mathfrak{p}$  indicates restriction to  $\mathfrak{p}$  of the linear

<sup>1</sup> Work supported in part by the National Science Foundation NSF GP-149.

transformation of  $\mathfrak{g}$  given by

$$\sinh ad X/ad X = \sum_{n \geq 0} (ad X)^{2n}/(2n + 1)!$$

Let  $D(G)$  and  $D(S)$  denote the set of left invariant (resp.  $G$ -invariant) differential operators on  $G$  (resp.  $S$ ). Let  $S(\mathfrak{a})$  denote the symmetric algebra over  $\mathfrak{a}$ ,  $\mathfrak{s}(\mathfrak{a})$  the space of  $C^\infty$  functions on  $\mathfrak{a}$  which are rapidly decreasing together with all their derivatives. Let  $I(\mathfrak{a})$  and  $\mathfrak{I}(\mathfrak{a})$  denote the set of  $W$ -invariants in  $S(\mathfrak{a})$  and  $\mathfrak{s}(\mathfrak{a})$ , respectively;  $\mathfrak{s}(\mathfrak{a})$  is taken with its usual locally convex topology [7, p. 90] and then  $\mathfrak{I}(\mathfrak{a})$  is a closed subspace. According to a theorem of Harish-Chandra (cf. [2, Theorem 1, p. 260], also [3, p. 432]) there exists an isomorphism  $\Gamma$  of the algebra  $D(S)$  onto  $I(\mathfrak{a})$ . Let  $I(G)$  denote the set of  $C^\infty$  functions  $f$  on  $G$  which are bi-invariant under  $K$  and for each integer  $q \geq 0$  and each  $D \in D(G)$  satisfy

$$\tau_{D,\mathfrak{a}}(f) = \sup_{H \in \mathfrak{a}} (1 + |H|)^q \omega(H) |(Df)(\exp H)| < \infty.$$

Let  $I_0(G)$  denote the space of functions of the form

$$\phi_{\mathfrak{a}}(x) = \int_{\mathfrak{a}} \pi(\lambda)a(\lambda)\phi_{\lambda}(x)d\lambda \quad (x \in G)$$

where  $a \in \mathfrak{s}(\mathfrak{a})$  and  $d\lambda$  is the Euclidean measure on  $\mathfrak{a}$ . Then, by [2, p. 586],  $I_0(G) \subset I(G)$  (it can be shown using [1] that  $\pi = \pi_0$ ). The seminorms  $\tau_{D,\mathfrak{a}}$  turn  $I_0(G)$  and  $I(G)$  into locally convex spaces.

LEMMA 1.  $I_0(G)$  is an algebra under convolution on  $G$ .

Under the restriction from  $G$  to  $A$ ,  $I_0(G)$  and  $I(G)$  are mapped isomorphically onto spaces  $I_0(A)$  and  $I(A)$  of  $W$ -invariant  $C^\infty$  functions on  $A$ . We carry the algebraic and topological structure of  $I_0(G)$  over on  $I_0(A)$  by means of this mapping. The space  $\mathfrak{I}(\mathfrak{a})$  is an algebra under convolution on  $\mathfrak{a}$ .

2. **Transmutation operators.** A function  $f$  on the space  $S=G/K$  is called a *radial function* if  $f(k \cdot p) = f(p)$  for all  $k \in K, p \in S$ . The set of continuous (resp.  $C^\infty$ ) radial functions on  $G/K$  is in one-to-one correspondence  $f \rightarrow \tilde{f}$  with the set of all continuous (resp.  $C^\infty$ )  $W$ -invariant functions on  $A$ . Here  $f(aK) = \tilde{f}(a)$  for  $a \in A$ . Let  $D \in D(S)$ ; then by [2, p. 265] there exists a unique differential operator  $\delta'(D)$  on  $A'$  (the set of regular elements in  $A$ ) such that  $(Df)^- = \delta'(D)\tilde{f}$  for all  $C^\infty$  radial functions  $f$ . The operator  $\delta'(D)$  is called the radial part of  $D$  in analogy with the radial part  $D_r^2 + (n - 1)/r D_r$  of the Laplacian on  $\mathbb{R}^n$ , ( $D_r = d/dr$ ). It is known [5] that there exists an isomorphism  $X$

(“transmutation operator”) of the vector space of even  $C^\infty$  functions on  $\mathbf{R}$ , onto itself, under which the singular operator  $D_r^2 + (n-1)/r D_r$  corresponds to  $D_r^2$ . The operators  $\delta'(D)$  ( $D \in \mathcal{D}(S)$ ) are singular when considered as differential operators on  $A$  but Theorem 1 shows that they have a simultaneous transmutation operator  $X$  under which they correspond to differential operators on the Euclidean space  $\mathfrak{a}$  with constant coefficients.

Given a  $W$ -invariant function  $\phi$  on  $A$ , let  $\tilde{\phi}$  denote the corresponding radial function on  $S$ . For  $\phi \in I(A)$ , put

$$(X\phi)(H) = e^{\rho(H)} \int_N \tilde{\phi}((\exp H)n \cdot o) dn \quad (H \in \mathfrak{a})$$

where  $dn$  is a suitably normalized invariant measure on  $N$ . As proved by Harish-Chandra [11, p. 595],  $X$  is a continuous mapping of  $I(A)$  into  $\mathfrak{g}(\mathfrak{a})$ .

**THEOREM 1.** *The mapping  $X$  is a topological isomorphism of the algebra  $I_0(A)$  onto the algebra  $\mathfrak{g}(\mathfrak{a})$ . Moreover, if  $D \in \mathcal{D}(S)$  then*

$$X\delta'(D)\phi = \Gamma(D)X\phi, \quad \phi \in I_0(A).$$

Here  $\Gamma(D)$  is considered as a differential operator on  $\mathfrak{a}$ .

The proof is based on the Plancherel formula for functions in  $I_0(G)$ , proved by Harish-Chandra [2]. It also uses the recent result of Gindikin and Karpelevič [1] according to which the function  $c(\lambda)$  above can be expressed in terms of Gamma functions.

**REMARKS.** At the end of [2], Harish-Chandra states the following two conjectures which would imply that  $I_0(A)$  contains all the  $W$ -invariant  $C^\infty$  functions on  $A$  with compact support.

(I) There exists a polynomial  $p \in S(\mathfrak{a})$  such that  $|c(\lambda)\pi(\lambda)p(\lambda)| \geq 1$  for all  $\lambda \in \mathfrak{a}$ . (Here we have used the fact that  $\pi = \pi_0$ .)

(II) The mapping  $X$  is one-to-one on  $I(A)$ .

Now (I) can be verified on the basis of the mentioned result of Gindikin and Karpelevič. Theorem 1 shows that (II) is equivalent to  $I_0(G) = I(G)$ . On the other hand, (II) is easily implied by the Plancherel formula for the functions in  $I(G)$ . This formula is not proved in [2] but I understand that Harish-Chandra has proved it in recent, as yet unpublished, work. In the next section we shall therefore assume that  $I_0(G) = I(G)$ .

**3. Fundamental solutions.** Let  $C_c^\infty(S)$  denote the space of  $C^\infty$  functions on  $S$  with compact support. Let  $\delta$  denote the distribution on  $S$

given by  $\delta(f) = f(o)$  where  $f \in C_c^\infty(S)$ . Let  $D \in \mathcal{D}(S)$ . A distribution  $T$  on  $S$  is called a *fundamental solution* of  $D$  if  $DT = \delta$ . If  $f \in C_c^\infty(S)$ , then a fundamental solution  $T$  of  $D$  gives a solution of the equation  $Du = f$  by putting  $u = f * T$  where  $*$  is the operation on distributions on  $G/K$  induced by the convolution product of distributions on  $G$ .

**THEOREM 2.** *Each invariant differential operator  $D \in \mathcal{D}(S)$  ( $D \neq 0$ ) on the symmetric space  $S$  has a fundamental solution.*

This is a consequence of Theorem 1 and the fact that a nonzero differential operator on  $\mathbf{R}^n$  with constant coefficients always has a tempered fundamental solution [4; 6].

*Added in proof.* In the case when  $G$  is complex the following formula (which is a simple consequence of Lemma 55 in [2]) gives a simpler proof of Theorem 2.

$$(DF) \circ \text{Exp} = \frac{1}{\omega} \lambda(D)(\omega(F \circ \text{Exp})) \quad (D \in \mathcal{D}(S)).$$

Here  $\text{Exp}$  is the usual Exponential mapping of  $\mathfrak{p}$  onto  $S$ ,  $\lambda$  is a certain isomorphism of  $\mathcal{D}(S)$  onto the algebra of  $\text{Ad}(K)$ -invariant polynomials on  $\mathfrak{p}$  and  $F$  is any radial function on  $S$ .

#### REFERENCES

1. S. G. Gindikin and F. I. Karpelevič, *Plancherel measure of Riemannian symmetric spaces of nonpositive curvature*, Dokl. Akad. Nauk. SSSR 145 (1962), 252–255 = Soviet Math. Dokl. 3 (1962), 962–965.
2. Harish-Chandra, *Spherical functions on a semisimple Lie group*. I, II, Amer. J. Math. 80 (1958), 241–310, 553–613.
3. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
4. L. Hörmander, *On the division of distributions by polynomials*, Ark. Mat. 3 (1958), 555–568.
5. J. L. Lions, *Équations différentielles opérationnelles*, Springer, Berlin, 1961.
6. S. Lojasiewicz, *Division d'une distribution par une fonction analytique de variables réelles*. C. R. Acad. Sci. Paris 246 (1958), 683–686.
7. L. Schwartz, *Théorie des distributions*. II, Hermann, Paris, 1951.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY