GENERIC AND RELATED MAPPINGS

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1. **Introduction.** An onto mapping $f(X)=Y$ will be called *generic* provided Y and X are homeomorphic topological spaces, or trivially, Y reduces to a single point. Thus such an f reproduces the topological structure of X provided it is not merely constant on X. The problem of determining conditions on f which will make it generic on a given type of space X has been under study for many years and, recently, has been the subject of intense interest, especially when X is a Euclidean space E^n or sphere S^n. Classically, the monotone mappings are generic when applied to the interval or circle and, by a well-known theorem of R. L. Moore, monotone nonseparating mappings are generic on the topological 2-sphere S^2 ("nonseparating" means that point-inverses do not separate the domain space X). For early discussions of the general problem and of most of the trial conditions to be mentioned below, see accounts of addresses by the author in [**4**] and the dissertation of Wardwell [**3**].

In this note a theorem will be proven in a general setting giving conditions under which a mapping which is restricted to a finite number of nondegenerate point inverses will be generic. It is then indicated how this result along with the Moore-Kline-Zippin-Bing sphere characterization readily yields the Moore theorem quoted above. Finally it is shown that various forms of pointlikeness and cellularity conditions on point-inverses are equivalent when each of these sets is interior to an n-cell in the domain space and, further, they are necessary conditions in this situation if the sets are isolated and the mapping is to be generic.

For simplicity all spaces used will be assumed metric. The interior of a subset Q of X will be denoted by int. Q. A mapping $f(X)=Y$ is *quasi-compact* if the image of every open inverse set is open [an inverse set K in X is one satisfying $K=f^{-1}f(K)$]. The *kernel* K_f of a mapping $f(X)=Y$ is the set of all $x\in X$ satisfying $x=f^{-1}f(x)$. That two sets X and Y are homeomorphic will be indicated by the symbol $X\overset{\text{top}}{\sim}Y$.

Given $X'\subset X$, $Y'\subset Y$, a homeomorphism $h(X')=Y'$ is a *strong homeomorphism* [**5**] provided that for any set $X_0\subset X'$ which is closed in X, $h(X_0)$ is closed in Y and for any $Y_0\subset Y'$ which is closed in Y, $h^{-1}(Y_0)$ is closed in X. Two subsets X' and Y' of X and Y are *strongly*

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homeomorphic, written $X' \sim^{\text{sth}} Y'$, provided there exists a strong homeomorphism of X' onto Y'.

A closed subset A of a space X is *stably pointlike* in X provided that if $\phi(X)=X'$ is a quasi-compact mapping of X with X' homeomorphic with X and A interior to the kernel of ϕ, then $X'-\phi(A)$ is strongly homeomorphic with $X'-a$ for some $a\in X'$, i.e., $X'-\phi(A) \sim^{\text{sth}} X'-a$ for some $a\in X'$. A closed set $A\subset X$ is simply *pointlike* (term due to Bing) provided $X-A\sim^{\text{sth}} X-a$ for some $a\in X$.

2. **General theorems.** We begin with the invariance result:

(2.1) THEOREM. *The property of being stably pointlike for a set K in X is invariant under any quasi-compact mapping f of X onto a homeomorph X' of X such that K is interior to the kernel of f.*

For let $f(X)=X'$ have K interior to its kernel. We have to prove that $f(K)$ is stably pointlike in X'. To this end let $\phi(X')=X_1$ be any quasi-compact mapping of X' onto a homeomorph X_1 of X' such that $f(K)$ is interior to the kernel of ϕ. Then

$$\phi f(X) = X_1$$

is a quasi-compact mapping of X onto X_1, X_1 is a homeomorph of X and K is interior to the kernel of this mapping ϕf. To see the latter, let U be an open set satisfying

$$K \subset U \subset \text{kernel of } f.$$

Then $f(U)$ is open by quasi-compactness of f. Hence there exists an open set V satisfying

$$f(K) \subset V \subset f(U)\cdot(\text{kernel of } \phi).$$

Whence

$$K \subset f^{-1}(V) \subset \text{kernel of } \phi f.$$

Accordingly, $X_1-\phi f(K)\sim^{\text{sth}} X_1-a$ for some $a\in X_1$, so that $f(K)$ is stably pointlike in X'.

(2.2) THEOREM. *If the mapping $f(X)=Y$ is quasi-compact and all point inverses are degenerate except for a finite number of points $b_1, b_2, \cdots, b_n\in Y$, then Y will be homeomorphic with X if each of the sets $A_1=f^{-1}(b_1), \cdots, A_n=f^{-1}(b_n)$ is stably pointlike in X.*

PROOF. By induction on n. If $n=1$, we have

(i) $X-A_1\sim^{\text{sth}} X-a_1$ for some $a_1\in X$,

(ii) $Y-b_1\sim^{\text{sth}} X-A_1$, since $X-A_1$ is the kernel of f.

Whence $X-a_1\sim^{\text{sth}} Y-b_1$ and this gives X homeomorphic with Y by a simple extension of a strong homeomorphism $h(X-a_1)=Y-b_1$ by defining $h(a_1)=b_1$.

Now assuming our conclusion established for $n=k-1$, we prove it for $n=k$. Let $\phi_1(X)=X_1$ be the natural mapping of the decomposition of X into A_1 and single points of $X-A_1$ as elements, and let $\phi_2(X_1)=X_2$ be the natural mapping of the decomposition of X_1 into the sets $\phi_1(A_2), \cdots, \phi_1(A_k)$ and single points of $X_1-\phi_1(A_2+\cdots+A_k)$ as elements. We have

(i) $$X_1 \sim^{\text{top}} X \text{ by case } n = 1 \text{ of our theorem.}$$

Also, since ϕ_1 is quasi-compact and has each of the sets $A_2, \cdots, A_k$ interior to its kernel, by (2.2) each of the sets $\phi_1(A_2), \cdots, \phi_1(A_k)$ is stably pointlike in X_1. Thus since ϕ_2 is quasi-compact, it follows by case $n=k-1$ of our conclusion that

(ii) $$X_2 \sim^{\text{top}} X_1 \sim^{\text{top}} X.$$

On the other hand since the mapping

$$g(X_2) = f\phi_1^{-1}\phi_2^{-1}(X_2) = Y$$

is (1-1) and quasi-compact it is a homeomorphism. This together with (ii) gives $Y\sim^{\text{top}}X$.

3. The 2-sphere theorem. We now indicate briefly how the case $n=2$ of (2.2) is adequate to yield the general theorem that *Monotone nonseparating mappings are generic on the* 2-*sphere* (R. L. Moore) by a simple argument.

To prove this let $f(X)=Y$ be monotone and nonseparating where X is a 2-sphere and Y is nondegenerate. Then Y is a locally connected continuum and has no cut point. By the Moore-Kline-Zippin-Van Kampen-Bing characterization of the 2-sphere, to prove Y a 2-sphere we must show:

(i) no pair of points separates Y, and

(ii) every simple closed curve on Y separates Y.

Now (i) follows directly from the facts that the 2-sphere X is unicoherent and unicoherence is invariant under all monotone mappings. Thus Y is unicoherent and no pair of its points can separate it since no single point can do so.

To prove (ii) let J be a simple closed curve in Y and let $R=Y-J$. We take distinct points $x, y\in J$ dividing J into open arcs A and B. Since each of the sets $X_0=f^{-1}(x)$ and $Y_0=f^{-1}(y)$ is a continuum not separating X, these sets are stably pointlike. Thus if $\phi(X)=X'$ is the natural mapping of X onto the quotient space X' obtained by decomposing X into the sets X_0, Y_0 and single points of $X-X_0-Y_0$, it follows by (2.2) that X' is homeomorphic with X and thus is a 2-sphere. By monotoneity of f the sets $A'=\phi f^{-1}(A)$ and $B'=\phi f^{-1}(B)$ are connected and their closures intersect in just the two points $x'=\phi(X_0)$ and $y'=\phi(Y_0)$. Hence there is a simple closed curve C in X' separating A' and B' and intersecting the closure of the union of these sets in just $x'+y'$. Now if R were connected, $\phi f^{-1}(R)$ would be connected and we could construct a θ-curve θ consisting of arcs $ax'b$ and $ay'b$ of C together with an arc $ab\subset\phi f^{-1}(R)$. However this is not possible because A' and B' would necessarily lie in the same component of $X'-\theta$ since each has $x'+y'$ in its closure. whereas $J\subset\theta$ and J separates A' and B' in X'.

4. **Equivalence and necessity of conditions.** A closed set A in a space X is *locally shrinkable* (see Wardwell [3] and compare with McAuley [2]) provided every open set about A contains an open set V about A such that there exists a point $a\in V$ and a strong homeomorphism of $X-A$ onto $X-a$ which is the identity on $X-V$. A closed set A in X is *cellular* (see Brown [1]) provided A is the intersection of a strictly decreasing sequence of n-cells in X for some n, i.e., each lies interior to its predecessor in the sequence. (It may be noted that a topological n-cell is not necessarily cellular in this sense, even in E^n.)

(4.1) THEOREM. *For a compact set A lying interior to an n-cell Q in X, the properties of being*: (a) *locally shrinkable*, (b) *stably pointlike in Q*, (c) *pointlike in Q, and* (d) *cellular, are equivalent.*

PROOF. (a) *implies* (b). For let $f(X)=X'$ be quasi-compact, $X'\sim^{\text{top}}X$, and so that $A\subset\text{int } K_f=K_0$. Let V be an open set about A chosen so that $\overline{V}\subset K_0$ and such that a point $a\in V$ and a strong homeomorphism h of $X-A$ onto $X-a$ exist with h fixed on $X-V$. It is then readily checked that fhf^{-1} is a strong homeomorphism of $X'-f(A)$ onto $X'-f(a)$.

(b) *implies* (c). Obvious.

(c) *implies* (d). Let $f(Q)=Q'$ be the natural mapping of the decomposition of Q into the set A and single points of $Q-A$. By (2.2), Q' is an n-cell and thus may be considered embedded in an n-sphere S^n. Since $A\subset\text{int } Q$ and A is the only nondegenerate point inverse for f, A is cellular by Theorem 3 of [1].

(d) *implies* (a). Essentially by Theorem 1 of [1].

(4.11) COROLLARY. *These equivalences hold, in particular, for arbitrary compact subsets of E^n or S^n.*

(4.2) THEOREM. *Any isolated nondegenerate point inverse A for a quasi-compact mapping $f(X)=Y$ which is interior to an n-cell Q in X and maps into a point a where Y is locally Euclidean has each of the properties* (a)–(d) *of* (4.1).

For if S is a sufficiently small sphere in Y with center a bounding a region R in Y about a so that $R+S\subset f$ (int $Q\cdot$int $K_f)+a$, it is clear that S is an S^{n-1}. Further, f^{-1} effects a topological embedding of a shell neighborhood of S into the interior of Q and thus effectively into the S^n one gets by mapping Q onto S^n so that the boundary of Q is the only nondegenerate point inverse. Thus by the Schoenflies theorem (see Brown [1]) the closure P of the component of $Q-f^{-1}(S)$ containing A is an n-cell. Since for the mapping $f: P\rightarrow R+S$, A is the only nondegenerate point inverse and since we may regard $R+S$ as embedded in an S^n, it follows by Theorem 3 of [1] that A is cellular.

(4.21) COROLLARY. *Any isolated nondegenerate point inverse for a quasi-compact mapping of E^n or S^n onto itself has properties* (a)–(d).

Added in proof. Indeed, this holds and the properties (a)–(d) remain equivalent for compact subsets of E^n or S^n, when the words "in Q" are omitted from the statements of properties (b) and (c). For it is a ready consequence of (4.2) that *If a pointlike set A in an n-manifold M^n lies interior to an n-cell Q in M^n, it is pointlike in Q.* To verify this, we let $f(M^n)=Y$ be the natural mapping of the decomposition of M^n into A and single points. Then Y is homeomorphic with M^n and thus is locally Euclidean at $f(A)$. Since A is the only nondegenerate point inverse, (4.2) applies and gives our conclusion.

In a less restricted setting a set may be pointlike but not stably pointlike. This is readily exhibited using a continuum constructed by Wardwell in [3] for a related purpose.

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