INFINITE MEASURE PRESERVING TRANSFORMATIONS WITH "MIXING"

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1. **Introduction.** It is well known that a transformation T which preserves a finite measure has the mixing property

$$(1.1)\qquad T^{(k)} = T \times T \times \cdots \times T \ (k \text{ times}, k \geqq 2) \text{ is ergodic}$$

$$\text{if and only if } T \text{ is weakly mixing [1].}$$

The purpose of this note is to give, for each positive integer k, an example of a transformation T which preserves a σ-finite infinite measure with the property,

$$(1.2)\qquad T^{(k)} \text{ is ergodic but } T^{(k+1)} \text{ is not ergodic.}$$

We also give an example of a transformation T which preserves a σ-finite infinite measure with the property

$$(1.3)\qquad T^{(k)} \text{ is ergodic for each } k = 1, 2, \cdots.$$

A transformation T with property (1.2) is said to have ergodic index k and a transformation T with property (1.3) is said to have infinite ergodic index. For completeness, we say that a nonergodic transformation has zero ergodic index.

Thus, for each $k = 0, 1, 2, \cdots, \infty$, infinite measure preserving transformations exist with ergodic index k, unlike finite measure preserving transformations which assume ergodic indices 0, 1, ∞ only.

The examples are taken from Gillis [2], and are Markov transformations derived from "centrally biased random-walks."

2. **Markov transformations preserving a σ-finite infinite measure.** Let

$$P = \|p(i,j)\|, \qquad i, j = 0, \pm 1, \pm 2, \cdots$$

be a stochastic matrix with only one ergodic class, i.e.,

$$p(i,j) \geqq 0, \qquad \sum_{j=-\infty}^{\infty} p(i,j) = 1,$$

and for each (i, j) there exists $n > 0$ for which $p^n(i, j) > 0$ where $P^n = \|p^n(i, j)\|$. Assume also that there exists a left eigenvector

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$\Lambda = \{\lambda(i)\}$ (with eigenvalue one) with positive entries such that

$$\sum_{i=-\infty}^{\infty} \lambda(i) = \infty.$$

Let Z be the set of all integers and let

$$X = \prod_{i=-\infty}^{\infty} Z_i, \qquad Z_i = Z, \qquad i = 0, \pm 1, \cdots.$$

A generic element of X is a point

$$x = \{z_i(x)\}.$$

A cylinder of X is a set of the form

$$C_{m,n}(x) = \{y \in X : z_i(x) = z_i(y),\ m \leqq i \leqq n\}.$$

Let $\mathcal{B}$ be the Borel field generated by the cylinders of X and let p be the σ-finite measure generated by the cylinder function

$$pC_{m,n}(x) = \lambda(z_m(x)) \prod_{i=m}^{n-1} p(z_i(x), z_{i+1}(x)).$$

It is clear that the measure p is invariant under the shift transformation T,

$$T\{z_i\} = \{z_i'\}, \qquad z_i' = z_{i+1},$$

and $X = \bigcup_{i=-\infty}^{\infty} X_i$, $p(X_i) = \lambda(i)$, $p(X) = \infty$, where $X_i = \{x \in X: z_0(x) = i\}$.

We refer to $(X, \mathcal{B}, p, T)$ as the σ-finite stationary Markov chain defined by P. T is the Markov transformation defined by P.

We shall be interested in the following conditions on P:

I_k. For every $i_1, \cdots, i_k; j_1 \cdots j_k$ there exists $n > 0$ such that

$$p^n(i_1, j_1) \times \cdots \times p^n(i_k, j_k) > 0.$$

II_k. $\sum_{n=1}^{\infty} [p^n(0,0)]^k = \infty$.

THEOREM. *P satisfies I_k and II_k if and only if the Markov transformation T defined by P satisfies: $T^{(k)}$ is ergodic with respect to $p^{(k)} = p \times \cdots \times p$ (k times).*

The above theorem can be deduced from a similar theorem in [**3**]. We indicate below the main points of the proof.

The theorem need only be proved for the case $k=1$. In fact, if

$$R(i_1, \cdots, i_k) = X_{i_1} \times \cdots \times X_{i_k},$$

then condition I_k states that

$$(2.1)\quad \lambda^{-1}(i_1)\cdots\lambda^{-1}(i_k)p^{(k)}[R(i_1\cdots i_k)\cap(T^{(k)})^{-n}R(j_1\cdots j_k)]>0$$

for some $n>0$. Condition II_k states that

$$(2.2)\quad \sum_{n=1}^{\infty}[\lambda(0)]^{-k}p^{(k)}[R(0,\cdots,0)\cap(T^{(k)})^{-n}R(0,\cdots,0)]=\infty.$$

The k-dimensional direct product $(X^{(k)}, \mathcal{B}^{(k)}, p^{(k)}, T^{(k)})$ of the system $(X, \mathcal{B}, p, T)$ can be regarded as 1-dimensional by relabelling the k-vector states $(i_1, \cdots, i_k)$ with integers. After relabelling, in view of (2.1) and (2.2) conditions I_k and II_k become I_1 and II_1.

If I_1 is not satisfied then for some (i,j), $p^n(i,j)=\lambda^{-1}(i)p(X_i\cap T^{-n}X_j)=0$ for all $n>0$ and T is not ergodic.

If II_1 is not satisfied then

$$\sum_{n=1}^{\infty}p^n(0,0)<\infty,$$

the state X_0 is not recurrent [4], and T is not ergodic since a wandering set of positive measure exists [1].

Suppose I_1 and II_1 are satisfied, then almost all points of X_0 return infinitely often to X_0 under both positive and negative iterations of T and the smallest invariant set containing X_0 is essentially the whole space X (cf. [4, §4]).

The remainder of the proof can be completed by showing that the transformation induced by T on X_0 [5], is a Bernoulli transformation. The ergodicity of T then follows from the ergodicity of the induced transformation [5].

3. **Examples.** Let $-1<\epsilon<1$, and define

$$Q=\|q(i,j)\|,\qquad i=0,\pm 1,\pm 2,\cdots$$

where $q(i, i+1)=(1-\epsilon/i)/2$, $q(i, i-1)=(1+\epsilon/i)/2$, $i\neq 0$, $q(0, 1)=q(0, -1)=1/2$, and $q(i,j)=0$ if $j\neq i+1$ and $j\neq i-1$.

Let $M=\{m(i)\}$, $i=0, \pm 1, \cdots$, where

$$m(0)=1,\quad m(i)=m(-i)=\frac{\Gamma(1+\epsilon)i\Gamma(i-\epsilon)}{\Gamma(1-\epsilon)\Gamma(i+1+\epsilon)},\qquad i>0.$$

One can easily verify that

$$MQ=M.$$

Let $Q^2=\|q^2(i,j)\|$ and put

$$P=\|p(i,j)\|.\qquad i,j=0,\pm 2,\pm 4,\cdots,$$

where $p(i,j)=q^2(i,j)$. Let $\Lambda=\{\lambda(i)\}$, $i=0, \pm 2, \pm 4$, where $\lambda(i)=m(i)$. Then $\Lambda P=\Lambda$ and $p(i, j)=0$ if and only if $j\neq i-2$, $j\neq i$ and $j\neq i+2$. P satisfies condition I_k for every $k=1, 2, \cdots$. (No difficulties arise from considering matrices P defined over the lattice of pairs of even integers.)

Moreover,

$$\sum_i \lambda(i) = \infty \qquad \text{if } -1 < \epsilon \leqq \frac{1}{2}$$

since

$$\lambda(n) \sim \frac{\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} n^{-2\epsilon}.$$

We shall need the following result of Gillis [2].

LEMMA. *For any $\theta>0$ there exists $K_1=K_1(\theta)$ such that for al' N,*

$$K_1^{-1}N^{\epsilon-1/2-\theta} < q^{2N}(0, 0) = p^N(0, 0) < K_1N^{\epsilon-1/2+\theta}.$$

Choose a positive integer k and $\eta>0$ such that

$$\frac{1}{k} > \frac{1+\eta}{1+k}.$$

Choose ϵ such that

$$\frac{1}{2} - \frac{1}{k} < \epsilon < \frac{1}{2} - \frac{1+\eta}{1+k}$$

and $\theta>0$ such that

$$\theta < \min\left(\epsilon - \frac{1}{2} + \frac{1}{k}, \frac{1}{2} - \epsilon - \frac{1+\eta}{1+k}\right);$$

then

$$-\frac{1}{k} < \epsilon - \frac{1}{2} - \theta < \epsilon - \frac{1}{2} + \theta < -\frac{1+\eta}{1+k}.$$

Consequently, by the lemma, there exists $K_1=K_1(\theta)$ such that

$$K_1N^{-1/k} < K_1N^{\epsilon-1/2-\theta} < p^N(0, 0) < K_1N^{\epsilon-1/2+\theta} < K_1N^{-(1+\eta)/(1+k)}$$

i.e.,

$$(p^N(0, 0))^{k+1} < (K_1)^{k+1}N^{-(1+\eta)}$$

and

$$(p^N(0, 0))^k > (K_1)^k N^{-1}.$$

Hence, by the theorem, the Markov transformation defined by P has ergodic index k.

Finally, if $\epsilon = 1/2$, then again by the lemma

$$\sum_{n=1}^{\infty} [p^n(0, 0)]^k = \infty \qquad \text{for } k = 1, 2, \cdots,$$

and consequently the Markov transformation defined by P has infinite ergodic index.

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