FLOWS ON SOLVMANIFOLDS

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Let G be a connected, simply connected solvable Lie group and let C be a closed subgroup such that G/C is compact. Further, let $g(t)$ be a one parameter subgroup of G. Then $g(t)$ induces, acting to the left on G/C, a one parameter group of transformations. We will call G/C a representation of a compact solvmanifold and the flow generated by $g(t)$ a G-induced flow. In [1] and [2] results concerning special G-induced flows were discussed in detail or research announced. It is the purpose of this note to state one aspect of the problem of G-induced flows and outline the solution that is now available. Full details and discussions will be presented elsewhere.

Let G/C be a representation of a compact solvmanifold. Our main problem may be stated as follows: Give a necessary and sufficient condition for the existence of an *ergodic* G-induced flow on G/C. We will now outline our solution to this problem.

Algebraic preliminaries. Let $L(G)$ denote a solvable Lie algebra which may be taken over the real or complex fields. Then $L(G)$ is said to be of Type (R) if all the roots of the algebra are either 0 or pure imaginary. One of the main algebraic facts we will need is the following:

THEOREM. *Given a solvable Lie algebra $L(G)$ there exists a unique minimal ideal H such that $L(G)/H$ is Type (R).*

We will also need the companion concept of a really regular element of $L(G)$. $X\in L(G)$ will be called really regular if it is

(1) regular,

(2) ad (X) has a maximal number of eigenvalues with nontrivial real part. The set of really regular elements is dense in $L(G)$.

THEOREM. *G/C has a G-induced ergodic flow if and only if there is a really regular element X of $L(G)$ such that* $\exp(tX)$ *induces an ergodic flow.*

REDUCTION THEOREM. *A really regular element in $L(G)$ induces an ergodic flow on G/C if and only if it induces an ergodic flow on* $\mathrm{Cl}(G/CH)$, *where H is the unique minimal analytic normal subgroup of G such that G/H is Type (R) and* Cl *denotes the closure operation.*

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Hence we see that we have reduced the problem to the Type (R) case.

More algebra I. Let S be a Type (R) solvable group, which is connected and simply connected. Then there exists a unique nilpotent analytic group N and a compact abelian group T of automorphisms of N such that

(a) $S \subset T \cdot N$.

(b) Let $p: S \rightarrow N$ be the projection mapping.

Then p is a homeomorphism of S onto N.

(c) S and N generate $T \cdot N$.

We will call $T \cdot N$ the minimal splitting of S.

More algebra II. Let C be a closed subgroup of a connected, simply connected Type (R) solvable group S. Let $T \cdot N$ be a minimal splitting for S. Then $S/N \cap C$ is compact and $C/N \cap C$ is a finite group. Further, the projection mapping $p: S \rightarrow N$ induces a homeomorphism of $S/N \cap C$ onto $N/N \cap C$.

DEFINITION. We will say that a one-parameter group in S is in general position if its projection onto T is dense in T, where $T \cdot N$ is the minimal splitting of S. Then one can easily see that the image of a really regular element of G is in general position in S and conversely, given an element of S in general position there exist really regular elements of G which project onto it.

SECOND REDUCTION THEOREM. *Let $S = G/H$, where H is the unique minimal analytic normal subgroup of G such that G/H is Type (R). A G-induced flow on S/C is ergodic if and only if there exists an S-induced flow which is ergodic on $S/N \cap C$.*

THEOREM. *Let S be of Type (R). There exists an S-induced ergodic flow on $S/N \cap C$ if and only if the null space of $(t - I)$, for all $t \in T$, contains a vector in general position in $N/C_0[N, N]$ relative to the lattice $C/C_0([N, N] \cap C)$, where C_0 is the identity component of C.*

REFERENCES

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