BOOK REVIEW

Topologische lineare Räume, Bd. 1. By Gottfried Köthe. (Die Grundlehren der Mathematischen Wissenschaften, Bd. 107.) Springer, Berlin, 1960. xii+456 pp.

It seems to be a commonplace nowadays that the concept of a vector space of finite or infinite dimension is an indispensable part of the education of every young mathematician whether pure or applied as well as of the theoretical physicist (or even the theoretical engineer, as some people already claim). Yet not too long ago, within the past forty years or so, this concept was unfamiliar or unknown (or at the very least not as fundamental as it appears today) to various very good mathematicians for whom a vector space was simply R^n or C^n. Hilbert spaces and later Banach spaces have progressively become ever more useful mathematical concepts, and geometrical intuition as applied to analysis has climbed from $n=1$, 2, 3, etc. to $n=\infty$. In this way, functional analysis became a well-established branch of mathematics. Even more recently, topological vector spaces as well have turned out to be basic to the degree that they have become a must in the education of any graduate student of mathematics.

At the present time, problems in analysis are treated on the dual basis of "hard" analysis, depending on the fine concepts and techniques of integration theory, analytic functions, differential equations, potential theory, harmonic analysis, etc. on R^n and C^n, and of the methods of "soft" analysis, namely those of Hilbert spaces, Banach spaces, topological vector spaces, their analogues for algebras and modules, etc. Of course, such a sharp boundary line between hard and soft analysis is tending to disappear. In some advanced mathematical communities it has indeed been forgotten, but some groups still stick to a firm distinction between these two tendencies as a matter of personal pride.

In the history of the ideas, methods, and applications of topological vector spaces up to the present time, there are two individuals whose names are landmarks in the development of this theory, namely Stephan Banach in connection with normed vector spaces and Laurent Schwartz in topological vector spaces. This is true not only because their work forms a very substantial contribution to the real progress of this branch of mathematics but also because they directly inspired many significant contributions in this area by their colleagues and students. It is interesting to read now what André Weil wrote about fifteen years ago in this respect: " . . . Le mathématicien ne pourra plus se contenter de l'espace de Hilbert, outil qui lui est devenu aussi familier que la série de Taylor ou l'intégrale de Lebesgue; est-ce dans la théorie des espaces de Banach qu'il faut chercher l'instrument approprié, ou faudra-t-il recourir à des espaces plus généraux? Il faut avouer que les espaces de Banach, pour intéressants et utiles qu'ils se soient déjà montrés, n'ont pas amené encore en analyse la révolution que certains en attendaient; mais ce serait jeter la manche après la cognée que d'en abandoner déjà l'étude, avant d'en avoir mieux exploré les diverses possibilités d'application. Peut-être cependant sont-ils à la fois trop généraux pour se prêter à une théorie aussi précise que celle de l'espace de Hilbert, et trop particuliers pour l'étude des opérateurs les plus intéressants. Par exemple, ils ne comprennent pas l'espace des fonctions indéfiniment dérivables; or, c'est seulement dans celui-ci qu'on peut définir les opérateurs de L. Schwartz, qui représentent formellement les dérivées de tout ordre de fonctions arbitraires; il y a là peut-être le principe d'un calcul nouveau, reposant en définitive sur le théorème de Stokes généralisé, et qui nous rendra accessibles les relations entre opérateurs différentiels et opérateurs intégraux." (See the article *L'avenir des mathématiques* in the volume *Les grands courants de la pensée mathématique*, Cahiers du Sud, Marseille, 1947.) It seems fair to say today that the elements of the theory of topological vector spaces play a role which is basic for teaching and research in analysis in the same sense that the elements of algebra and topology are fundamental for the whole of mathematics.

The book under review, written by a leading figure in the field of functional analysis, is a most valuable contribution to the literature on topological vector spaces. We should mention immediately the fact that N. Bourbaki's *Espaces vectoriels topologiques* is a recent, up-to-date, and well-chosen presentation of what supposedly (with some exaggeration) every mathematician ought to know about topological vector spaces. Accordingly Bourbaki's EVT does not include or develop in its text several aspects of the theory which experts in the field or other mathematicians looking for information would hope to find in explicit form. An analogous remark applies to A. Grothendieck's *Espaces vectoriels topologiques*. The present book was written, it appears, with the two-fold purpose of being a textbook which can be read by graduate students (who will even find in it an opening chapter on point set topology) as well as a reference work for research mathematicians (who will find in it a wealth of information).

In the latter direction, it constitutes the first treatise devoted to the various aspects of topological vector spaces.

Chapter 1 presents in its fifty pages the fundamental notions of point set topology needed in the sequel, such as topological spaces, directed systems, filters, compact spaces, metric spaces, uniform spaces, and real functions. The chapter contains a coherent exposition of only the minimum of notions and results used in the book, but it is clearly written and can serve as a text for a graduate course on point set topology, for it has the advantage of being well organized, self-contained and not prolix.

Chapter 2 deals with vector spaces over any field, linear mappings, matrices, algebraic duals, tensor products and linear equations. This chapter is not austere linear algebra, since it contains an exposition on linear topology in a vector space over a discrete field, a notion systematically studied by Lefschetz. One would almost say that this topology is algebraic as it finds its inspiration and application mostly to algebraic situations.

Chapter 3 starts the treatment of real or complex topological vector spaces. Banach spaces, topological vector spaces, the Banach-Schauder and Banach-Steinhaus theorems on open linear mappings, the closed graph theorem, equicontinuity and equivalent or related concepts, are discussed at length, together with convex sets, their separation, and the various versions of the Hahn-Banach theorem. This is a rather classical and basic part of the theory, which has been standard material in the teaching of functional analysis.

Chapter 4 constitutes a good exposition of topological vector spaces which are locally convex and therefore includes all the topological vector spaces, real or complex, of mathematics, except for a few isolated or pathological exceptions. Fréchet spaces are introduced. Inductive and projective limits (already known in other branches of mathematics and introduced in the theory of locally convex spaces by Dieudonné and Schwartz, in view of their necessity in the theory of distributions and spaces of holomorphic functions) are presented in a comprehensive way. This is an important but as yet not widely known aspect of the theory. The chapter also includes the strong topology, Mackey's topology, the Banach-Dieudonné theorem and a rather extensive discussion of the fundamental notion of duality, as well as its relations to the main techniques of constructing subspaces, quotient spaces, product spaces, direct sums, projective and inductive limits.

Chapter 5 contains the all-important notions of reflexive and semireflexive spaces, the theorems of Smulian, Eberlein, Pták and Krein-Milman, as well as a study of metric properties of normed spaces (exemplified by strict convexity, smallest distances, weak or strong differentiability of the norm, uniform convexity and other aspects) which is mostly addressed to specialists.

In the final Chapter 6, the author gets down to certain special classes of locally convex spaces, such as barreled spaces ("espaces tonnelés" of Bourbaki), Montel spaces, bornological spaces, (DF) spaces in the sense of Grothendieck and Köthe spaces (which are not named by the author after himself, contrary to standard usage, but instead are still called vollkommene Räume in the author's traditional terminology). The chapter ends with several examples in order to illustrate possible pathologies in this context.

The text is very well written and rather ambitious in bringing together in book form such a wealth of topics which are either fundamental and should be grasped by every graduate student; or important and should be known by the specialists; or amusing and should be read by people who want to discard false conjectures. As is no surprise for anyone who knows the author personally, the book is very up to date in results, terminology and taste. In this respect, it has been strongly influenced by Bourbaki and the French school. Needless to say, the author has succeeded in reaching his aims and has provided us with a valuable source of information.

Any reviewer has supposedly the duty of pointing out that the author did not include in his presentation of the field various contributions which ought at least to be mentioned. It is not reasonable to make this kind of remark in the present case since the book under review is Vol. 1 of a series of $n \geqq 1$ volumes. For a number of years, it has been known that the author was writing this text. Let us hope that Professor Köthe will find some spare time from his duties as the Rector of the University of Heidelberg in order to finish in a reasonably finite length of time the treatise on topological vector spaces, which he has so successfully begun.

There is a natural question that a thorough exposition, like the one at hand, brings to mind: Now that the general linear methods in mathematics which are discussed in this book are nearing completion, what will their non-linear analogues be?

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