ON THE THEORY OF AUTOMORPHIC FUNCTIONS AND THE PROBLEM OF MODULI[1]

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The subject of this presentation is the problem of moduli for certain special types of algebraic-geometrical objects and the connection of this problem with the theory of automorphic functions for certain arithmetically defined discontinuous groups acting on bounded domains in complex Euclidean space.

To begin with, the earliest such situation of nontrivial content was the theory of elliptic modular functions, connected with the theory of moduli of elliptic curves. This situation was subsequently generalized to the theory of Siegel's modular functions connected with the moduli of normally polarized Abelian varieties. The theory of moduli of algebraic curves or of compact Riemann surfaces of genus n, which has received independent development from the analytical point of view by Teichmüller [9], Ahlfors [1], and Bers [4], may also be extracted to a large degree from the theory of moduli of normally polarized Abelian varieties of dimension n, from among which, by certain algebraic-geometrical criteria due to Matsusaka [6], may be distinguished those which are the canonically polarized Jacobian varieties of curves of genus n. The essential point is that the Jacobian varieties form a Zariski-open subset of an algebraic subset of the normally polarized Abelian varieties [2]. (Of course here we should be careful to point out the distinction between results of a more algebraic nature on the theory of moduli, and the results discussed here which are of a transcendental nature, linking such a theory of moduli with the theory of automorphic functions of some discontinuous group.)

The fact that we speak only of the moduli of polarized Abelian varieties comes of course from the well-known phenomenon that if, in the space of $n \times 2n$ period matrices of complex, n-dimensional tori, we identify those points which correspond to complex analytically isomorphic tori, what we get is a space which is not even Hausdorff, not to speak of being a complex space or algebraic variety. A polarization for a complex torus T is a class consisting of all divisors D

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on T which are numerically equivalent to the positive rational multiples of some positive, nondegenerate divisor D_0 on T, or equivalently, is a class of positive rational multiples of some principal form for the period matrix of T (the concept of polarization is due to Weil [**10**]). The prescription of a polarization not only cuts down the size of the space we consider, but also the size of the equivalence classes within that space, the equivalence classes being the points of the space of moduli. Naturally, T has a polarization only if T is an Abelian variety, and T may have more than one polarization only if its ring of endomorphisms has some special properties.

What we are concerned with here to begin with is the general problem of moduli of polarized Abelian varieties whose endomorphism ring contains some preassigned ring (of course only of certain admissible types) in a way which we shall now explain.

Let A be an Abelian variety and $\mathfrak{C}$ a polarization of A. Denote by $\mathfrak{A}(A)$ and $\mathfrak{A}_0(A)$ respectively the endomorphism ring of A and its tensor product with the rational numbers Q, the algebra of endomorphisms. $\mathfrak{C}$ induces a positive involution $*$ of $\mathfrak{A}_0(A)$. Let L be some algebra of finite dimension over Q with involution ρ. If ϕ is an isomorphism of L into $\mathfrak{A}_0(A)$ such that $\phi(x^\rho)=\phi(x)^*$, we say that $(A, \mathfrak{C})$ is of type (L, ρ, ϕ).

One problem treated by Shimura and others is that of incorporating a polarized Abelian variety $(A, \mathfrak{C})$ of given type (L, ρ, ϕ) into a maximal complex analytic fiber system of such polarized varieties of the given type. By an analytic family of Abelian varieties, we mean a triple $(\mathfrak{A}, \lambda, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are irreducible complex analytic spaces and λ is a proper complex analytic mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ having the following properties:

(i) There is an analytic subset $\mathfrak{E}$ of $\mathfrak{B}$ such that for $b\in\mathfrak{B}-\mathfrak{E}$, the fiber $A_b=\lambda^{-1}(b)$ is an Abelian variety of fixed dimension n.

(ii) We define $\mathfrak{A}^{(l)}=(\lambda^l)^{-1}$ (diagonal of B^l) and $\lambda^{(l)}=\lambda^l|\mathfrak{A}^{(l)}$ (for any positive integer l); then the group law of A_b is cut out on $A_b^{(3)}=\lambda^{(3)-1}(b)$ by a fixed analytic subset $\mathfrak{G}$ of $\mathfrak{A}^{(3)}$ for all $b\in\mathfrak{B}-\mathfrak{E}$.

If in addition $\mathfrak{r}$ is some ring with unit 1, we say that $(\mathfrak{A}, \lambda, \mathfrak{B})$ admits $\mathfrak{r}$ as an endomorphism ring if we have further:

(iii) For each $\rho\in\mathfrak{r}$ we are given an analytic subset $\iota(\rho)$ of $\mathfrak{A}^{(2)}$ such that for each $b\in\mathfrak{B}-\mathfrak{E}$, $\iota_b(\rho)=\iota(\rho)\cap A_b^{(2)}$ is the graph of an endomorphism of A_b and such that

$$\iota_b\colon \rho\to\iota_b(\rho),\quad b\in\mathfrak{B}-\mathfrak{E},$$

is an isomorphism of $\mathfrak{r}$ into $\mathfrak{A}(A_b)$, and $\iota_b(1)$ is the identity on A_b.

In general the problem of constructing an analytic family of polarized Abelian varieties $(A, \mathfrak{C})$ of type (L, ρ, ϕ) makes sense only when one prescribes some order in L. The usual method of constructing such analytic families may be indicated by the following example: Denote by H_n the space of $n\times n$ symmetric complex matrices $Z=X+iY$, such that Y is positive definite. Let L be a totally real algebraic number field of degree d over the rational numbers Q. By a lattice in L^m we mean a free Abelian subgroup M of L^m having md independent generators. Let $Z=(Z_1, \cdots, Z_d)\in H_n^d$, let E denote the $n\times n$ identity matrix, and denote by M a lattice in L^{2n}. Denote by M_Z the lattice in C^{nd} consisting of all vectors of the form

$$((Z_1E)\xi^{(1)}, \cdots, (Z_dE)\xi^{(d)}), \qquad \xi\in M,$$

where (Z_jE) denotes an $n\times 2n$ matrix decomposed into $n\times n$ blocks, and $\xi^{(i)}=\xi^{\sigma_i}$, $i=1, \cdots, d$, denote the conjugates of ξ over Q, $\sigma_1, \cdots, \sigma_s$ denoting the distinct isomorphisms of L into the reals (ξ, etc., are column vectors). C^{nd}/M_Z is an Abelian variety, and the skew-symmetric form defined on pairs (x, y) of d-tuples of $2n$-dimensional real vectors $x=(x_1, \cdots, x_d)$, $y=(y_1, \cdots, y_d)$ by $S(x, y) = \sum_1^d {}^tx_iJy_i$, where

$$J=\begin{pmatrix} 0 & E\\ -E & 0\end{pmatrix},$$

defines a principal form for M_Z. We define an equivalence relation on $H_n^d\times C^{nd}$ by $(Z, \zeta)\sim(Z', \zeta')$ if and only if $Z=Z'$ and $\zeta-\zeta'\in M_Z$. Then the quotient $\mathfrak{A}$ of $H_n^d\times C^{nd}$ by this equivalence relation, together with $\mathfrak{B}=H_n^d$ and the projection of $\mathfrak{A}$ onto $\mathfrak{B}$ is a fiber system of Abelian varieties ($\mathcal{E}$ is the empty set) which has any subring of the order of M in L as a ring of endomorphisms, the mapping ι being defined in an obvious manner. This fiber system may be called typical in the sense that any other analytic fiber system of Abelian varieties with a given polarization, of given dimension, of given type (L, ρ, ϕ), and admitting the given ring $\mathfrak{r}=\text{ord}(M)$ as endomorphism ring is locally induced by a fiber system of the type we have just described. Moreover, if we denote by Γ_M the subgroup of those $g\in\text{Sp}(n, L)$ such that ${}^tgM=M$, then Γ_M operates naturally on H_n^d, and the orbit space $V=H_n^d/\Gamma_M$ has a natural compactification V^*, which is a projective variety, on which V is a Zariski-open set [3]. Moreover, if $r=\text{ord}(M)$ is the maximal order in L (i.e., the ring of all integers in L), V^* is defined over Q, and if $x\in V$, then $Q(x)$ is the field of moduli of the corresponding Abelian variety with endomorphism ring $\mathfrak{r}$. All this can be proved using the theory of Satake compactifications [7] and the theory of θ-functions.

H_n^d is of course the Hermitian symmetric space associated with the semi-simple Lie group $\mathrm{Sp}(n, R)^d$, and the latter may be viewed in an evident way as the scalar extension of $\mathrm{Sp}(n, L)$ by R over the rational numbers. The group Γ_M is commensurable with the group $\mathrm{Sp}(n, \mathfrak{o})$ consisting of those elements of $\mathrm{Sp}(n, L)$ with elements in the ring of integers $\mathfrak{o}$.

We now give a discussion along the lines of Shimura to cover a more general case of the above. Let L be a division algebra of finite dimension over Q with positive involution ρ. For any positive integer m, we may extend ρ to $M_m(L)$ (the ring of $m \times m$ matrices over L) by defining $(l_{ij})^\rho = (l_{ji}^\rho)$. Suppose $(A, \mathfrak{C})$ is an Abelian variety of dimension n and of type (L, ρ, ϕ). Choose a lattice D in C^n such that A is isomorphic to C^n/D. If $\phi(1)$ is the identity endomorphism of A, then $[L:Q]$ divides $2n$ and we have $2n = [L:Q]m$. By appropriate choice of coordinates in C^n, $D \otimes_Z Q$ may be written in the form $\{ \sum_{i=1}^m \phi(a_i)x_i \mid (a_1, \cdots, a_m) \in M \}$, for some suitable lattice M in L^m. Let $\mathfrak{r}$ be the order of M in L. Then a principal form for D may be identified with a ρ-skew-symmetric element T of $M_m(L)$ such that $\mathrm{tr}(M^\rho TM) \subset Z$. T defines an involution σ of $M_m(L)$ by $\sigma(m) = T^{-1}m^\rho T$. Let $G_{\sigma,Q}$ denote the subgroup of the group $\mathrm{GL}(m, L)$ consisting of those g which commute with σ, or in other words the group of ρ-units of T. Let G_σ be the extension of $G_{\sigma,Q}$ to the reals over the rational numbers. Then G_σ is a semi-simple Lie group, $G_\sigma = G_1 \times \cdots \times G_g$, and if K_j is a maximal compact subgroup of G_j, then G_j/K_j is a Hermitian symmetric space D_j realizable as a bounded domain in the space of some number of complex variables. Let

$$\Gamma_M = \{ g \in G_{\sigma,Q} \mid g^\rho M = M \}.$$

Γ_M is a discontinuous group on D, and the points of D/Γ_M may be identified with the isomorphism classes of similarly polarized Abelian varieties of the same type (L, ρ, ϕ) giving rise to the same M. The above discussion is essentially Shimura's [8]. The actual possibilities for L have been found by Albert and are as follows:

I. L a totally real number field F. Then each D_i may be identified with the space of $m/2 \times m/2$, complex, symmetric matrices $Z = X + iY$, with Y positive definite (denoted by $Y > 0$), and $G_i = \mathrm{Sp}(m/2, R)$.

II. A central simple division algebra L over F, a totally real algebraic number field, such that all simple components of $L \otimes_Q R$ are isomorphic to $M_2(R)$. G_i and D_i are the same as in I with $m/2$ replaced by m.

III. A central simple division algebra L over F such that all simple components of $L \otimes_Q R$ are isomorphic to the real quaternions K. In this case D_i is the space of $m\times m$, complex, skew-symmetric matrices Z such that $E-Z^t\bar{Z}>0$, and G_i is the group of quaternion units (quaternion orthogonal group) of a skew-quaternionic-Hermitian form.

IV. A central simple division algebra L over a purely imaginary quadratic extension F_0 of F. Then D_i may be identified with the space of $r_i\times s_i$ complex matrices Z such that $E_{r_i}-Z^t\bar{Z}>0$ (here r_i and s_i are suitable positive integers such that $r_i+s_i=mq$, where $[L:F_0]=q^2$), and G_i is the Hermitian orthogonal group of

$$\begin{pmatrix} E_{r_i} & 0 \\ 0 & -E_{s_i} \end{pmatrix}.$$

In order to obtain a satisfactory transcendental theory of moduli for such Abelian varieties, it is essential to obtain, if possible, a compactification of $D/\Gamma_M=V$ as a complex analytic space V^* realizable as a complex projective variety. Such a result has already been obtained for the groups commensurable with the Hilbert-Siegel modular group [3] (Case I above).

Using results of Borel [5] or Satake [7] we can in fact obtain a little more general result than this. Namely, let L be a division algebra of finite dimension over Q with involution ρ, and extend ρ to $M_m(L)$ as above. Let T be a ρ-symmetric or ρ-skew-symmetric, ρ-sesqui-linear form on L^m, and let $G_{\sigma,Q}$ be the group of ρ-orthogonal matrices with respect to T in $\mathrm{GL}(m, L)$. Extend $G_{\sigma,Q}$ to G_σ as before; then G_σ is a semi-simple real Lie group $G_\sigma=G_1\times G_2\times\cdots\times G_d$. Let K_i be a maximal compact subgroup of G_i. We explicitly assume that each of the spaces G_i/K_i is Hermitian symmetric of classical type, and hence equivalent to one of the four types of classical bounded domains in Cartan's classification, and we also assume each $D_i=G_i/K_i$ is of the same type, I, II, III, or IV. Put $D=D_1\times\cdots\times D_d$. Let M be a lattice in L^m and let $\Gamma_M=\{g\in G_{\sigma,Q}\mid g^\rho M=M\}$. Then if Γ is any subgroup of G_σ commensurable with Γ_M, the orbit space D/Γ has a natural compactification V^*, V^* is a complex analytic space, and as such may be realized as a projective algebraic variety.

The methods of proving this are analogous to those used in the case where Γ is the Siegel modular group or the Hilbert-Siegel modular group. We begin by observing that the Γ-rational boundary components are in correspondence with the totally isotropic, L-rational spaces of T in L^m. For a given dimension q, we denote by $\mathfrak{F}_1,\cdots,\mathfrak{F}_t$ a complete system of Γ-inequivalent, Γ-rational boundary components of D of dimension q, and by $\mathfrak{L}_{\mathfrak{F}_i}$ the module of Poincaré

θ-series of sufficiently high weight m on $\mathfrak{F}_i$. We can define a Φ-operator analogous to that in the case of the Siegel modular group, and we can define series analogous to Eisenstein series which make it possible to prove that if $\mathfrak{L}$ is the module of modular forms of some suitable high weight on D, then $\Phi(\mathfrak{L}) \supset \Pi_i \mathfrak{L}_{\mathfrak{F}_i}$. This assures us of the existence of enough local "holomorphic" functions on V^* to prove it is a complex space, and of enough modular forms on V^* to imbed it in some complex projective space.

What one actually would like to obtain is a similar result in case we are given, say, a simple algebraic group defined over Q such that if G_i is any of the simple factors of G_R, then G_i/K_i is Hermitian symmetric—in this case we consider discontinuous groups contained in G_R which are commensurable with G_Z, the group of integral unimodular matrices in G_R. Of course this means one must deal with two exceptional groups as well as with the classical ones. Moreover, one would also like to discover in which of such cases one can attach a meaningful theory of moduli for some algebraic-geometrical structures to the orbit spaces of the corresponding discontinuous groups.

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