PERIODIC TRAJECTORIES OF A ONE-PARAMETER SEMIGROUP

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The object of Theorem 1 below is to establish the existence of periodic solutions of an autonomous differential equation $\dot{y}=f(y)$ by an extension of the Poincaré method of sections (see [**2**; **4**]). The following situation is envisaged: the equation is defined on a subset D of euclidean space and has unique solutions $y(x, t)$ jointly continuous in t and the initial point x; D contains a compact subset K with the property that the positive trajectories starting from points of K remain in K. The assignment to x in K and t in $[0, \infty)$ of the point $T_t(x)=y(x, t)$ in K defines a continuous one-parameter semigroup T_t acting on K, i.e., T_t is jointly continuous in x and t, T_0 is the identity on K and $T_{s+t}=T_s \circ T_t$.

THEOREM 1. *Let K be a connected finite complex, let T_t be a continuous one-parameter semigroup acting on K and let ω be a closed* 1*-form on K (defined over a portion of euclidean space containing K) with integer-valued periods. Make the following two assumptions on K, T_t and ω:*

A. *For each x in K there is a t for which the integral of ω over the trajectory from x to $T_t(x)$ is positive.*

B. *The classes of closed paths in K over which the integral of ω vanishes form a subgroup of the fundamental group of K. Assume that the corresponding covering space K^i has nonvanishing Euler characteristic.*

Conclusion: T_t has a periodic trajectory, i.e., there is an x in K and a period $p>0$ such that $T_{t+p}(x)=T_t(x)$ for all $t\geqq 0$.

REMARK a. If we denote the integral of ω over the trajectory from x to $T_t(x)$ by $\Delta(x, t)$, assumption A implies that there exists a positive constant a such that $at<\Delta(x, t)$ for sufficiently large t. Thus $\Delta(x, t)$ converges uniformly to $+\infty$. If T_t is engendered by the differential equation $\dot{y}=f(y)$, $\Delta(x, t)$ can be written as the integral with respect to t of the scalar product $\omega\cdot f$, evaluated along the trajectory from x to $T_t(x)$.

REMARK b. Although the covering space K' is not a finite complex, assumption A implies that K' has finite Betti numbers, so that its Euler characteristic is defined.

REMARK c. The period of the periodic trajectory disclosed by the theorem is bounded by a number depending on the uniform rate of convergence of $\Delta(x, t)$ to $+\infty$, the periods of ω and the Betti numbers of K'.

By means of the construction outlined in [2], Theorem 1 can be derived from the following theorem.

THEOREM 2. *Let F be an upper semicontinuous multiple-valued function from a finite complex X into itself. Let the system of endomorphisms F_{*p} of $H_p(X)$, the homology groups of X with real coefficients, be induced by F. Denote by r_p the lowest value to which rank F_{*p}^k descends as k increases. Then $\sum(-1)^p r_p \neq 0$ implies that F has a periodic point: $x \in F^N(x)$ and the period N does not exceed the larger of $\sum r_{2q}$ and $\sum r_{2q+1}$.*

The proof of Theorem 2, using the Lefschetz formula for multiple-valued functions [4; 5] is essentially the same as that of the more special theorem in [1].

The relationships in Theorem 1 are illustrated by the following construction. Let f be any continuous mapping of a connected finite complex X into itself. The mapping cylinder C_f of f, constructed using two copies of X, one for the domain and one for the range of f, can be made into a space K by identifying the two copies. A semigroup T_t acting on K is obtained by moving all points at a uniform rate along the segments from x to $f(x)$. A closed 1-form ω with integer periods can be defined on K which is zero on the subspace X and such that the integral of ω over any segment from x to $f(x)$ is $+1$; ω satisfies assumption A. The covering space K' is then a space obtained by coupling together copies C_f^n, $-\infty < n < +\infty$, of C_f. For the endomorphism f_{*p} of $H_p(X)$, the integer r_p defined in the statement of Theorem 2 turns out to be the pth Betti number of K', so that by Theorem 2 nonvanishing of the Euler characteristic of K' (assumption B) implies that f has a periodic point and T_t a periodic trajectory.

A proof of Theorem 1 will appear elsewhere.

REFERENCES

1. F. B. Fuller, *The existence of periodic points*, Ann. of Math. (2) **57** (1953), 229–230.

2. ———, *Fixed points of multiple-valued transformations*, Bull. Amer. Math. Soc. **67** (1961), 165–169.

3. S. Lefschetz, *Differential equations: geometric theory*, Interscience, New York, 1957.

4. ———, *On coincidences of transformations*, Bol. Soc. Mat. Mexicana **2** (1957), 16–25.

5. B. O'Neill, *Induced homology homomorphisms for set-valued maps*, Pacific J. Math. **7** (1957), 1179–1184.

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