A REDUCTION OF THE POINCARÉ CONJECTURE TO OTHER CONJECTURES. II

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In a recent note I formulated two conjectures [2, p. 365], which imply the Poincaré conjecture. Bernard Maskit produced a counter example to the second of those conjectures [1]. In the present note I modify those conjectures, in a way suggested by his counter example and his two examples. The modified conjectures imply the Poincaré conjecture and avoid the obstacles raised by his examples. To avoid confusion, in the remainder of this note, I will be referring to the counter example of Maskit as the first example of [1] and to the other ones as the second and third examples of [1] respectively.

In this note we use the notations and conventions of [2]. In the first example of [1] the word $b_1[b_1^{-1}, b_2]$ is *not* cyclically reduced but, as an element of F, is representable by a *simple* loop on N. In the second example of [1] the word $b_1[b_2^{-1}, a_1^{-1}]$ is *cyclically reduced* but, as an element of F, is *not* representable by a simple loop on N. This can be seen by using the algorithm of Bruce L. Reinhart [4, §3]. In the third example of [1] the word $b_1[b_2, a_2]$ is *cyclically reduced* and moreover, as an element of F, is representable by a *simple* loop on N.

Before we express the modified conjectures, let us compare the first and third examples of [1]. In the first example A_1', see [2, p. 365, ll. 3–9], is obtained through a transformation of B_1 by B_2. In the third example we first have to change our "coordinate system" on N, and consider a new system A_i'', B_i'', $i=1$, 2, such that B_1'' or B_2'' represents the element $b_1[b_2, a_2]$ or $b_2[b_1, a_1]$ of F. We then observe that A_i is obtained through a transformation of A_i''. Thus the first and third examples of [1] have a common feature. Namely, the first is obtained by transforming B_1, while the third is obtained by transforming A_1 (and changing the coordinate system). This common feature suggests that, we have to "pass to the geodesics," before expressing the conjectures.

Let N be an orientable closed surface of genus $p \geqq 2$, and let A_i, B_i, $i=1, \cdots, p$, be a *fundamental system* of N with base point o. As is well known, N has certain hyperbolic metrics, imposed on it by Poincaré, and therefore there are geodesics on N, see [3, No. 3].

Let now G_i and H_i be the primary closed geodesics on N, cor-

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responding to A_i and B_i respectively, $i=1, \cdots, p$ [**3**, Lemma (12.2)]. Then G_i and H_i have only one point q_i in common [**3**, Nos. 3 and 12], any two of the pairs G_i, H_i different from each other are disjoint, and if we cut N along the G and H's we obtain a 2-sphere with p holes having boundaries $G_iH_iG_i^{-1}H_i^{-1}$.

Let G_i', $i=1, \cdots, p$, be *primary simple closed geodesics* on N, any two different ones being *disjoint*, and such that G_i' is *homologous* to H_i on N. Let o_i be *some* common point of G_i and G_i'.

Let now j be a fixed natural number $\geqq 1$ and $\leqq p$. By an isotopic deformation of each one of the pairs G_i, H_i, $i=1, \cdots, p$, on N, such that q_j moves on G_j and q_i moves to o_j, we obtain a new fundamental system A_{ji}, B_{ji}, $i=1, \cdots, p$, on N with base point o_j, such that the carrier of A_{jj} is precisely G_j. Then

$$\pi_1(N, o_j) \approx F_j = \left(a_{j1}, b_{j1}, \cdots, a_{jp}, b_{jp}\colon \prod_{i=1}^{p} [a_{ji}, b_{ji}]\right),$$
$$j = 1, \cdots, p \geqq 2.$$

Let $\bar{\Delta}_j$ be the smallest normal subgroup of F_j, containing the element $[a_{jj}, c_{jj}]$, where c_{jj} is the element of F_j corresponding to the loop obtained from the geodesic G_j'. Let finally $\overline{D}_j \to N$ be the regular covering corresponding to $\bar{\Delta}_j$. We observe that the group of covering translations is $F_j/\bar{\Delta}_j$.

Under the above hypotheses, the modified conjectures are the following.

(1) *For any* $p \geqq 2$, *there exists a* j ($\geqq 1$ *and* $\leqq p$), *such that the group* $F_j/\bar{\Delta}_j$ *has no elements of finite order.*

(2) *For the same* j *as above,* $\overline{D}_j$ *is planar.*

The Conjectures (1) and (2) imply the Poincaré conjecture. The method is precisely that described in [**2**, Nos. 1–4], with the only difference that, at the end we "pass to the geodesics" G_i, H_i and G_i' of A_i, B_i and A_i' respectively, $i=1, \cdots, p$, see [**2**, p. 365, ll. 3–9 and ll. 8–6 from the bottom] and [**3**, Lemma (12.2)].

Let now Φ_j be the free group freely generated by $a_{j1}, b_{j1}, \cdots, a_{jp}, b_{jp}$. We observe that $c_{jj} = b_{jj}\tau_j$, where $\tau_j \in [\Phi_j, \Phi_j]$. Then

$$\bar{\Delta}_j = \langle [a_{jj}, b_{jj}\tau_j] \rangle$$

$$F_j/\bar{\Delta}_j \approx \left(a_{j1}, b_{j1}, \cdots, a_{jp}, b_{jp}\colon \prod_{i=1}^{p} [a_{ji}, b_{ji}], [a_{jj}, b_{jj}\tau_j]\right).$$

Therefore our Conjectures (1) and (2) above are essentially special cases of [**2**, p. 365, Conjectures (5.1) and (5.2)]. That the last group has no elements of finite order has been proved recently by Elvira S. Rapaport [5].

The Conjectures (1) and (2) avoid the obstacles raised by the examples of [1]. Actually, let us consider the first (or third) example of [1], we note that $p=2$. Let G_1 and G_1' be the closed geodesics on N corresponding to the words a_1 and $b_1[b_1^{-1}, b_2]$ (or a_1 and $b_1[b_2, a_2]$) respectively. Then G_1 and G_1' are primary and simple [3, Lemma (12.2)], and they have only one point o_1 in common. Thus there is a *simple* loop L on N, which is homotopic on N to

$$L_1L_1'L_1^{-1}(L_1')^{-1}$$

where L_1 and L_1' are the loops with base point o_1, obtained from the geodesics G_1 and G_1' respectively. Therefore $F_1/\bar{\Delta}_1$ has no elements of finite order, and $\overline{D}_1$ is planar [3, Lemma (8.1)]. Hence the first and third examples of [1] are no counter examples to the above Conjectures (1) and (2). As far as the second example of [1] is concerned, this shows that it is indispensable that the geodesics G_i', $i=1, \cdots, p$, be simple, see [2, p. 365, ll. 8–6 from the bottom].

It was the first example of [1] which compelled me to modify the conjectures, from [2, p. 365, Conjectures (5.1) and (5.2)] to another form, where the property "$b_j\tau_j$ being cyclically reduced" comes into play. Finally, it was the third example of [1] which compelled me to modify the conjectures once again, and give them the above form (1) and (2).

Concluding the present note we would like to emphasize that, $\bar{\Delta}_j$ depends on the point o_j, $j=1, \cdots, p$. Thus the proper selection of the *index* j, and of the *point* o_j, may affect the validity of the Conjecture (2).

Bibliography

1. Bernard Maskit, *On a conjecture concerning planar coverings of surfaces*, Bull. Amer. Math. Soc. **69** (1963), 396–398.

2. C. D. Papakyriakopoulos, *A reduction of the Poincaré conjecture to other conjectures*, Bull. Amer. Math. Soc. **68** (1962), 360–366.

3. ———, *A reduction of the Poincaré conjecture to group theoretic conjectures*, Ann. of Math. (2) **77** (1963), 250–305.

4. Bruce L. Reinhart, *Algorithms for Jordan curves on compact surfaces*, Ann. of Math. (2) **75** (1962), 209–222.

5. Elvira S. Rapaport, *On a problem of Papakyriakopoulos*, Bull. Amer. Math. Soc. **69** (1963), 402–404.

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