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A NONTRIVIAL NORMAL SUP NORM ALGEBRA

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Let X be a compact Hausdorff space and A a complex linear algebra of continuous complex-valued functions defined on X. Suppose A is *normal* on X, i.e., for every pair of disjoint closed sets K_0, K_1 in X, there exists a function $f \in A$ such that $f(K_0)=0$ and $f(K_1)=1$. Does it follow that every continuous complex-valued function on X can be uniformly approximated by functions in A? With the additional assumption that A is closed under complex conjugation, it follows by the Stone-Weierstrass theorem. (Trivially, if A is normal then A separates points.) The same theorem implies that the analogous question in the case of real-valued functions has an affirmative answer. However, in the complex-valued case it need not be so. An example will be given which demonstrates this. In this example, the space X is a suitably chosen compact set in the complex plane. The algebra is $R(X)$, the algebra of all functions which can be uniformly approximated on X by rational functions whose poles lie outside X. It will be shown that $R(X)$ is normal on X and is a proper sub-algebra of $C(X)$, the algebra of all continuous complex-valued functions on X. Since $R(X)$ is closed under uniform limits, this will be sufficient.

Two lemmas are needed to accomplish this. One is a modification of an observation of Mergelyan [1]. The second represents a slight extension of a result due to Beurling [2].

[1] The research in this paper forms part of the author's doctoral dissertation, submitted to the Massachusetts Institute of Technology, Spring, 1963.

LEMMA 1. *Let* $\{\Delta_i\}$ *be a sequence of open discs in the complex plane such that* $\sum_i$ *(radius* Δ_i*)* <1. *If*

$$X = \{z: |z| \leqq 1\} - \bigcup_{i=1}^{\infty} \Delta_i$$

then $R(X) \neq C(X)$.

PROOF. It suffices to demonstrate the existence of a finite complex Baire measure μ on X which is orthogonal to every function in $R(X)$ and is not the zero measure.

Let M be the space of finite complex Baire measures on the closed unit disc. Let

$$X_n = \{z: |z| \leqq 1\} - \bigcup_{i=1}^{n} \Delta_i.$$

The boundary of X_n, ∂X_n, is the sum of a finite number of arcs of circles. Define $\mu_n \in M$ to be the measure which coincides with dz on ∂X_n and is zero elsewhere. Since the length of ∂X_n is less than 4π, the total variation of μ_n certainly satisfies $\|\mu_n\| \leqq 4\pi$. Thus the sequence $\{\mu_n\}$ has a subsequence which converges in the weak-star topology on M to a measure μ. It is easily verified that μ is supported on X, the intersection of the X_n.

Let r be a rational function whose poles lie outside X. There exists a positive integer $N = N(r)$ such that the poles of r lie outside X_N. By the Cauchy integral theorem

$$\int_{|z|\leqq 1} r\,d\mu_n = \int_{\partial X_n} r\,dz = 0, \qquad \text{for } n \geqq N.$$

Since μ is the weak-star limit of a subsequence of $\{\mu_n\}$

$$\int_{|z|\leqq 1} r\,d\mu = \int_X r\,d\mu = 0.$$

This implies that μ is orthogonal to $R(X)$.

Let A_n be the area of X_n and A the area (Lebesgue measure) of X. By Green's theorem,

$$\int_{|z|\leqq 1} \bar{z}\,d\mu_n = \int_{\partial X_n} \bar{z}\,dz = 2iA_n.$$

Therefore,

$$\int_{|z|\leqq 1} \bar{z}\,d\mu = \int_X \bar{z}\,d\mu = 2i \lim_{n\to\infty} A_n = 2iA.$$

But

$$A \geqq \pi\left[1 - \sum_i (\text{rad } \Delta_i)^2\right] \geqq \pi\left[1 - \sum_i (\text{rad } \Delta_i)\right] > 0.$$

Thus μ is not the zero measure.

LEMMA 2. *Let D be an open disc in the complex plane. For any $\epsilon>0$, there exists a sequence $\{\Delta_k\}$ of open discs contained in D and a sequence $\{r_n\}$ of rational functions such that:*

(i) $\sum_k (\text{rad } \Delta_k) < \epsilon$.

(ii) *The poles of r_n lie in $\bigcup_{k=1}^n \Delta_k$.*

(iii) *The sequence $\{r_n\}$ converges uniformly on the complement of $\bigcup_{k=1}^\infty \Delta_k$ to a function which is identically zero outside D and is nowhere zero on $D - \bigcup_{k=1}^\infty \Delta_k$.*

PROOF. It suffices to prove this when D is the open unit disc. The proof, as mentioned earlier, rests heavily on a construction due to Beurling [2]. This construction will now be described.

The infinite product

$$\phi(z) = \prod_2^\infty (1 - (e^{1/\log n} z)^n)$$

converges uniformly in any disc $|z| \leqq R < 1$. Let $\alpha_1, \alpha_2, \cdots, \alpha_k, \cdots$ be the zeroes of $\phi(z)$ arranged so that $|\alpha_k| \leqq |\alpha_{k+1}|$. Let m_R be the minimum value of $|\phi(z)|$ on the circle $|z| = R$. As Beurling proved, there is a sequence of circles $\{|z| = R_n\}$ such that m_{R_n} tends to infinity as R_n tends to one. Also, for k sufficiently large, the derivative $\phi'(z)$ satisfies

$$\left|\frac{1}{\phi'(\alpha_k)}\right| < e^{-\sqrt{k}}.$$

Using these properties of $\phi(z)$, it is established by straightforward application of the Cauchy integral theorem that

$$\sum_{k=1}^\infty \frac{1}{\phi'(\alpha_k)(z - \alpha_k)} = \frac{1}{\phi(z)} \qquad \text{for } |z| < 1,\ z \neq \alpha_k$$

and

$$\sum_{k=1}^\infty \frac{1}{\phi'(\alpha_k)(z - \alpha_k)} \equiv 0 \qquad \text{for } |z| \geqq 1.$$

Given this result of Beurling, it remains to show that a sequence

$\{\Delta_k\}$ of open discs centered at the α_k can be chosen so that the series

$$\sum_{k=1}^{\infty} \frac{1}{\phi'(\alpha_k)(z-\alpha_k)}$$

converges uniformly on the complement of $\bigcup_{k=1}^{\infty} \Delta_k$. Moreover each Δ_k must be contained in D and condition (i) must be satisfied.

A simple computation shows that, for k sufficiently large, an open disc centered at α_k having radius $1/k^2$ will be contained in D. Let N be a positive integer such that this condition, as well as

$$\left| \frac{1}{\phi'(\alpha_k)} \right| < e^{-\sqrt{k}},$$

is satisfied when $k \geqq N$. Require also that

$$\sum_{N}^{\infty} \frac{1}{k^2} < \frac{\epsilon}{2} \cdot$$

Such an N certainly exists.

For $k \geqq N$, choose Δ_k so that

$$(\text{rad } \Delta_k) = \frac{1}{k^2} \cdot$$

For $1 \leqq k < N$, choose Δ_k so that it is contained in D and

$$(\text{rad } \Delta_k) < \frac{\epsilon}{2N} \cdot$$

If z is any point in the complement of $\bigcup_{k=1}^{\infty} \Delta_k$ then

$$\left| \sum_{N}^{\infty} \frac{1}{\phi'(\alpha_k)(z-\alpha_k)} \right| \leqq \sum_{N}^{\infty} \left| \frac{1}{\phi'(\alpha_k)(z-\alpha_k)} \right| \leqq \sum_{N}^{\infty} k^2 e^{-\sqrt{k}} < \infty .$$

This establishes uniform convergence on the complement of $\bigcup_{k=1}^{\infty} \Delta_k$. q.e.d. Lemma 2.

Let $\{D_m\}$ be the sequence of all open discs in the plane centered at points with rational co-ordinates and having rational radii. For each m, in accordance with Lemma 2, there exists a sequence $\{\Delta_k^m\}$ of open sub-discs of D_m such that

$$\sum_{k} (\text{rad } \Delta_k^m) < \frac{1}{2^m} \cdot$$

(Apply Lemma 2 letting $D = D_m$, $\epsilon = 1/2^m$.)

Consider now the sequence of open discs in the plane consisting of the Δ_k^m as m and k range over all positive integers. Clearly

$$\sum_{m,k} (\text{rad } \Delta_k^m) < 1.$$

Let

$$X = \{z: |z| \leqq 1\} - \bigcup_{m,k} \Delta_k^m.$$

By Lemma 1, $R(X)$ is a proper sub-algebra of $C(X)$. It remains to show that $R(X)$ is normal on X.

Let k be any closed set in X and p any point in $X-k$. There is an m such that $p \in D_m$ and $D_m \cap k = \emptyset$. Since the Δ_k^m were chosen in accordance with Lemma 2, there exists an $f \in R(X)$ such that $f(p) \neq 0$ and $f(k) = 0$. If a commutative Banach algebra with identity has this separability property on its maximal ideal space, i.e., separates an arbitrary closed set and a point not in that set, it is normal on its maximal ideal space [**3**, Lemma 25C]. But for any compact X in the plane, $R(X)$, with the sup norm

$$\|f\| = \sup_{x \in X} |f(x)|,$$

is a commutative Banach algebra with identity and it is well known that the maximal ideal space of $R(X)$ is X.

Much of the credit for this example belongs to Professor Kenneth Hoffman of the Massachusetts Institute of Technology. It was his conjecture originally that an example of this kind could be found and it was he who supplied many important ideas along the way. I wish to thank him for this. My thanks also go to Professor John Wermer of Brown University who first realized the significance of Beurling's result, from a function algebra point of view, and directed my attention to it.

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